Open Science Journal of Statistics and Application
2019; 6(2): 20-27
http://www.openscienceonline.com/journal/osjsa
ISSN: 2381-4950 (Print); ISSN: 2381-4969 (Online)

Modified Iterative Algorithm for Solving Optimal Control Problems

Maha Delphi, Suha Shihab*

Department of Applied Sciences, University of Technology, Baghdad, Iraq

Email address
alrawy1978@yahoo.com (S. Shihab)
*Corresponding author

To cite this article
Maha Delphi, Suha Shihab. Modified Iterative Algorithm for Solving Optimal Control Problems. *Open Science Journal of Statistics and Application.* Vol. 6, No. 2, 2019, pp. 20-27.

Received: May 22, 2019; **Accepted**: July 3, 2019; **Published**: July 16, 2019

Abstract

In this paper, the study of problems in optimal control is very important in our day life and their applications can be studied in many disciplines based on mathematical modeling physics, chemistry and economy. Because of the complexity of most applications, optimal control problems are solved numerically. New techniques for achieving an approximate solution to optimal control problems are considered. They are based upon B-spline polynomials approximation with state parameterization method. New useful property of B-spline polynomials is first derived then, it is utilized to propose a modified restarted technique to reduce the number of unknown parameters with fast convergence. Furthermore, it can be proved that with special knot sequence, the B-spline basis are exactly Bernstein polynomials. The objective of the present work is to propose an approximate technique for solving linear and nonlinear optimal control problems is presented. The algorithm modifies previous works to certain optimal control problems and is depended on a Bernstein series expansion of state parameterization. The differential expressions from the constraint and the cost index as well as the boundary conditions are reduced into algebraic equations. The technique starts from initial trajectory is based on the boundary conditions then new iterative method with the help Bernstein polynomials and produces satisfactory convergence with small number of unknown parameters. The applicability of the proposed algorithm is illustrated on four linear and nonlinear optimal control problems. The comparison with other works is also included in this paper.

Keywords

Bernstein Polynomials (BEPs), Optimal Control Problems (OCPs), Parameterization Technique

1. Introduction

The study of problems in optimal control is very important in our day life. The application of optimal control problems can be studied in many disciplines based on mathematical modeling physics, chemistry, and economy [1-3]. Because of the complexity of most applications, optimal control problems are solved numerically. Various numerical methods have been proposed to solve (OCPs). In [4] Yousef Edrisi studied the solution of OCPs using collocation method using B-spline functions. Authors of [5] presented a numerical solution of OCPs with aid of state parameterization technique. Different numerical algorithms for treating OCPs have been introduced by utilizing the orthogonal functions. The complexity of the OCPs is decreased by reducing it to an algebraic system of equation, for example, B-spline polynomials [6], generalized Laguerre polynomials [7], Chebyshev polynomials [8-10] as well as third kind Chebyshev wavelets functions [11], Boubaker polynomials [12]. Special attention is given to find the approximate solution of OCPs using BEPs. These polynomials have already been utilized for solving OCPs [13] and integral equation [14]. In [15], authors have constructed orthonormal BEPs and applied them to solve integral equations.

The approach in the current paper based on BEPs expansion for solving OCPs. These polynomials introduced by [16-17]. In [17], Mohson, applied the operational matrices of BEPs and proposed a numerical solution of fractional optimal control problems while Safaie and Farahi in [16] solved delay fractional OCP with the aid of BEPs. For the historical development of BEPs properties and their applications, the reader can be referred to [19-20].

2. Bernstein Polynomials: Definition and Properties

2.1. Definition of BEPs

The general form of BEPs of degree in n of the interval (0, 1) is defined by:

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i} \; 0 \le i \le n$$

where $\binom{n}{i} = \frac{n!}{i!(n-i)!}$.

That is:

a) linear BEPs $B_i^1(t)$ where:

$$B_0^1(t) = 1 - t, B_1^1(t) = t$$

b) Quadratic BEPs $B_0^2(t)$ where:

$$B_0^2(t) = (1-t)^2, B_1^2(t) = 2t(1-t), B_2^2(t) = t^2$$

c) Cubic BEPs $B_i^3(t)$ where:

$$B_0^3(t) = (1-t)^3, B_1^3(t) = 3t(1-t)^2, B_2^3(t) = 3t^2(1-t), B_3^3(t) = t^3$$

For mathematical convenience, $B_i^n(t)$ is equal to Zero if $i < 0$ or $i < n$.

The derivative of the n^{th} degree BEPs are polynomials of degree $n-1$ and are given by:

$$DB_i^n(t) = n(B_{i-1}^{n-1}(t) - B_i^{n-1}(t)) \quad (1)$$

where $D \equiv \frac{d}{dt}$.

it is important in numerical formulation of the problem using the basis for $n \ge 1$ with the following useful degree elevation property:

$$B_{i-1}^{n-1}(t) = \frac{1}{2}[(n-i)B_i^n(t) + (i+1)B_{i+1}^n(t)] \quad (2)$$

The values of BEFs at the end points are:

$$B_i^n(0) = \begin{cases} 1, i = 0 \\ 0, i = 1, 2, \dots, n \end{cases} \text{ and } B_i^n(1) = \begin{cases} 0 \; i = 0, 1, \dots, n-1 \\ 1 \; i = n \end{cases}$$

2.2. Function Approximation

A square integrable functions $f(t)$ in $(0, 1)$ can be expressed in terms of the BEPs basis:

$$f(t) = \sum_{c=0}^{n} c_i B_i^n(t) = c^T Q(t) \quad (3)$$

where $c^T = [c_0, c_1, \dots, c_n]$and $Q(t) = [B_0^n(t), B_1^n(t), \dots, B_n^n(t)]^T$.

2.3. New Property of BEPs

The following proposition presents the relationship between BEPs and the power of t.

Proposition: the power basis $\{t^i\}_{i=0}^{s}$ can be rewritten in terms of Bernstein through the following relation:

$$t^n = [B_{n-1}^{n-1}(t) - \frac{1}{n} B_{n-1}^n(t)] \quad (4)$$

For $n = 2,3, \dots . i$, where $1 = B_{01} + B_{11}$ $t = B_{11}$.

Proof:

In order to establish the validity of this proposition since we have:

$$B_k^n(t) = \sum_{i=k}^{n} (-1)^{i-k} \binom{n}{1} \binom{i}{k} t^i$$

therefore;

$$B_{n-1}^{n-1}(t) = \sum_{i=n-1}^{n-1} (-1)^{i-n+1} \binom{n-1}{i} \binom{i}{n-1} t^i$$

or $B_{n-1}^{n-1}(t) = -1^{n-1-n+1} \binom{n-1}{n-1} \binom{n-1}{n-1} t^{n-1}$:

$$\text{that is } B_{n-1}^{n-1}(t) = t^{n-1} \tag{5}$$

Now,

$$B_{n-1}^{n}(t) = \sum_{i=n-1}^{n} (-1)^{i-n+1} \binom{n}{i} \binom{i}{n-1} t^i$$

$$= \binom{n}{n-1} \binom{n-1}{n-1} t^{n-1} - \binom{n}{n-1} t^n$$

$$\text{That is } B_{n-1}^{n}(t) = nt^{n-1} - nt^n \tag{6}$$

From (5) and (6) one can get:

$$B_{n-1}^{n}(t) - \frac{1}{n} B_{n-1}^{n}(t) = t^{n-1} - \frac{1}{n}(nt^{n-1} - nt^n)$$

Or:

$B_{n-1}^{n-1}(t) - \frac{1}{n} B_{n-1}^{n}(t) = t^n$ which is the required result.

3. Outline of the Method

3.1. The Problem Statement

Find the optimal control $u(t)$ which minimizes the cost function:

$$J = \int_0^1 F\big(t, x(t), u(t)\big) dt \tag{7}$$

Subject to:

a) state equation defined by:

$$u(t) = f(t, x(t), \dot{x}(t) \tag{8}$$

where $x(.) : [0,1] \to R$ is the state variable.

$u(.) : [0,1] \to R$ is the control variable.

f: is a real valued continuously differentiable function.

b) initial condition $x(0) = x_0$ and final condition:

$$x(1) = x_1 \tag{9}$$

where x_0 and x_1 are states given in R.

3.2. Solution Scheme

A robust technique for finding an approximate solution to optimal control problem in this subsection.

First we start with the following approximation by:

$$x^1(t) = (a_0 B_0^1(t) + a_1 B_1^1(t)) \tag{10}$$

Using the initial condition: $x(0) = a_0 B_0^1(0) + a_1 B_1^1(0)$

final condition: $x(1) = a_0 B_0^1(1) + a_1 B_1^1(1)$

Eq. (4) leads to $x_0 = a_0$ and $x_1 = a_1$.

After substituting these values into eq (10) one can get:

$$x^1(t) = x_0 B_0^1(t) + x_1 B_1^1(t)$$

or $x^1(t) = x_0 + (x_1 - x_0)t$.
The optimal control u (t) can be obtained from eq. (8) to obtain:

$$u^1(t) = f(t, x^1(t), \dot{x}^1(t)) \tag{11}$$

The functional J can be evaluated using eq. (7) to get:

$$J^1 = \int_0^1 F(t, x^1(t), u^1(t)dt$$

Now the next approximation $x^2(t), u^2(t), J^2(t)$ is calculated in the next step as below:

$$x^2(t) = x(t)^1 + a_2[-\frac{1}{2}B_1^2(t)$$

$$u^2 = F(t, x^2(t), \dot{x}^2(t))$$

$$J^2 = \int_0^1 F(t, x^2(t), u^2(t))$$

By continuing the procedure, the n[th] approximated solution for $x(t)$, $u(t)$ will be as follows:

$$x^{n+1}(t) = x^n(t) + a_{n+1}[B_n^n(t) - \frac{1}{n+1}B_n^{n+1}(t) - B_{n-1}^{n-1}(t) + \frac{1}{n}B_{n-1}^n(t)]$$

$$u^{n+1} = F(t, x^{n+1}(t), \dot{x}^{n+1}(t))$$

$$J^{n+1} = \int_0^1 F(t, x^{n+1}(t), u^{n+1}(t)dt$$

It is useful to give the following algorithm which summarizes the proposed method.
The algorithm:
For the accuracy of the solution choose $\in > 0$.
Step 1: Let $n = 1$ and put $x(t)^1 = x^0(t) + (x^1 + x^0)B_1^1(t)$, $\omega^1(t) = J^1(x^1(0))$.

$$x^2(t) = x^1(t) + a_2[-\frac{1}{2}\, B_1^2(t)]$$

and $\omega^2(t) = J(x^2(.))$.
Step 2: let $n = 2 \to n + 1$.

$$x^{n+1}(t) = x^{n-1}(t) + a_{n+1}[\, B_n^n(t) - \frac{1}{n+1}\, B_n^{n+1}(t) - \, B_{n-1}^{n-1}(t) + \frac{1}{n}\, B_{n-1}^n(t)]$$

and $\omega^n(t) = J(x^n(0))$.
Step 3: If $|\omega^n - \omega^{n-1}| < \in$ then stop, otherwise go to step 2.

$$\omega^n(t) = J(x^n(.))$$

4. Application Examples

The following examples are considered to illustrate the efficiency of the proposed algorithm.
Example (1)
This example clarifies the following concepts:
Find the optimal state and optimal control based on minimizing the performance index:

$$\mathrm{J} = \int_0^1 \left(x(t) - \frac{1}{2}u(t)^2\right) dt\,, 0 \le t \le 1$$

subject to $u(t) = \dot{x}(t) + x(t)$ with the condition $x(0) = 0$, $x(1) = \frac{1}{2}(1 - \frac{1}{e})^2$.
The exact solution for the state $x(t)$ and the control $u(t)$ is:

$$x(t) = 1 - 0.5e^{t-1} - 0.8160603e^{-t}$$

$$u(t) = 1 - e^{t-1}$$

and $J_{exact} = 0.08404562020$.

Start with the initial approximation to be $x^1(t) = 0.199788 * B_1^1(t)$.

The optimal state and control variables can be approximated with the aide of our algorithm are achieved as:

For $n = 1$: $x^2(t) = 0.199788B_1^1(t) + 0.204561B_1^2(t)$.

$$u^2(t) = 0.608927B_0^1(t) - 0.009563B_1^1(t) + 0.20451B_1^2(t)$$

For n=2: $x^3(t) = 0.188337B_1^1(t) + 0.210397B_1^2(t) + 0.011654B_2^2(t) -$

$$0.003885B_2^3(t)$$

$$u^3(t) = 0.608927B_0^1(t) - 0.044323B_1^1(t) + 0.192915B_1^2(t) + 0.046617B_2^2(t) - 0.003885B_2^3(t)$$

For$n = 3$: $x^4(t) = 0.188134B_1^1(t) + 0.210397B_1^2(t) + 0.009018B_2^2(t) - 0.003006B_2^3(t) + 0.002636B_3^3(t) - 0.000659B_3^4(t)$

$$u^4(t) = 0.608927B_0^1(t) - 0.052435B_1^1(t) + 0.196826B_1^2(t) + 0.054526B_2^2(t) - 0.038156B_2^3(t) + 0.002636B_3^3(t) - 0.000659\ B_3^4(t)\ b88B_3^4(t)$$

The approximate results are obtained by the proposed algorithm with $n = 1, 2, 3$ and compared our results by results of Mehne [21]. Our results have almost better accuracy. Total information are listed in Table 1 which illustrates the optimal values for the functional J with different iterations.

Table 1. *Results of the functional J.*

Iteration	Our method	Error	Mehne method [21]	Error
1	0.08401526011	3.047×10^{-5}	0.05332622101	3.0×10^{-2}
2	0.08402489318	2.073×10^{-5}	0.0840152600	3.0×10^{-2}
3	0.08402519637	2.0424×10^{-5}	0.8402496180	2.0×10^{-2}

Note that, the optimum values a_1, a_2, a_3 for minimizing the functional J are:

$$a_2 = -0.409139,\ a_3 = 0.011654, \text{ and } a_4 = 0.002636$$

The primacy of present algorithm compared with Mehme method is clear in this example because by the same number of iteration n, the present algorithm error are lower.

Example (2)

Consider the non-linear control system which consists of minimizing:

$$\int_0^1 u^2(t)dt$$

subject to $u(t) = \dot{x(t)} - x^2(t)sint$, $x(0) = 0$, $x(1) = 0.5$.

The initial approximation in this example is:

$$x^1(t) = 0.5 * B_1^1(t)$$

For $n = 1$.

$$x^2(t) = 0.5B_1^1(t) - 0.0189B_1^2(t)$$

$$u^2(t) = 0.4622B_0^1(t) + 0.5378B_1^1(t) - \sin(1)[0.5B_1^1(t) - 0.0189B_1^2(t)]^2$$

For $n = 2$.

$$x^3(t) = 0.4837B_1^1(t) - 0.01075B_1^2(t) + 0.0163B_2^2(t) - 0.005433B_2^3(t)$$

$$u^3(t) = 0.4622B_0^1(t) + 0.5052B_1^1(t) - 0.02445B_1^2(t) + 0.0489B_2^2(t) - \sin(t)[0.4837B_1^1(t) - 0.01075B_1^2(t) + 0.0163B_2^2(t) - 0.005433B_2^3(t)]^2$$

For $n = 3$.

$$x^4(t) = 0.4837B_1^1(t) - 0.01075B_1^2(t) + 2.1159B_2^2(t)$$

$$-0.7053B_2^3(t) - 2.0996B_3^3(t) - 0.5249B_3^4(t)$$

$$u^4(t) = 0.4622B_0^1(t) + 6.804B_1^1(t) - 3.17385B_1^2(t)$$

$$-8.3495B_2^2(t) + 2.799467B_2^3(t)$$

$$-\sin(t)[0.4837B_1^1(t) - 0.01075B_1^2(t) + 2.1159B_2^2(t) - 0.7053B_2^3(t) - 2.0996B_3^3(t) - 0.5249B_3^4(t)]^2$$

The optimum values a_1, a_2 and a_3 for minimizing the functional J are:
$a_2 = 0.0378$, $a_3 = 0.0163$., and $a_4 = -2.0996$.
The value of $J = 0.2005$ for $n = 1, 2, 3$.
Example (3)
The proposed method in this example is applied to the following problem.

$$J= \frac{1}{2}\int_0^1 (3x(t)^2 + u(t)^2)dt$$

subject to $u(t) = \dot{x(t)} + x(t)$, $x(0) = 0$, $x(1) = 2$.
The initial approximation in this example is $x^1(t) = 2 * B_1^1(t)$.
For $n = 1$: $x^2(t) = 2B_1^1(t) - 0.714286B_1^2(t)$.

$$u^2(t) = 0.571429B_0^1(t) + 5.428571B_1^1(t) - 0.714286B_1^2(t)$$

For $n = 2$.

$$x^3(t) = 1.611111B_1^1(t) - 0.519841B_1^2(t) + 0.388889B_2^2(t) - 0.129621B_2^3(t)$$

$$u^3(t) = 0.571429B_0^1(t) + 3.261511B_1^1(t) - 1.103041B_1^2(t)$$

$$+1.555289B_2^2(t) - 0.129621B_2^3(t)$$

For $n = 3$.

$$x^4(t) = 1.611111B_1^1(t) - 0.519841B_1^2(t) + 0.179812B_2^2(t)$$

$$-0.059937B_2^3(t) + 0.209077B_3^3(t) - 0.052269B_3^4(t)$$

$$u^4(t) = 0.571429B_0^1(t) + 3.634211(t) - 0.789391B_1^2(t)$$

$$+2.183612B_2^2(t) - 0.059937B_2^3(t) - 0.069723B_3^3(t) - 0.052269B_3^4(t)$$

The optimum values a_1, a_2 and a_3 for minimizing the functional J are $a_2 = 1.4280$, $a_3 = 0.3888$, $a_4 = 0.2091$. The comparison among the BEPs algorithm with n=1, 2, 3 beside Mehne method [21] are listed in Table 2 and the exact value for the cost is J=6.1586.

***Table 2.** The values of cost functional J in example 3.*

Iteration	Our method	Error	Mehne method [21]	Error
1	6.19047619	0.0318762	6.1905	0.0319
2	6.177513228	0.01891323	6.1775	0.0189
3	6.174827155	0.016227	6.1753	0.0167

Example (4)
The proposed method in this example is applied to the following problem:

$$J= \int_0^1 (x(t)^2 + u(t)^2)dt$$

subject to $u = \dot{x}$, $x(0) = 0$, $x(1) = 0.5$.
The initial approximation in this example is:

$$x^1(t) = 0.5 * B_1^1(t)$$

For $n = 1$.

$$x^2(t) = 0.5B_1^1(t) - 0.056818B_1^2(t)$$

$$u^2(t) = 0.386364B_0^1(t) + 0.613636B_1^1(t)$$

For $n = 2$.

$$x^3(t) = 0.470833B_1^1(t) - 0.042235B_1^2(t) + 0.029167B_2^2(t) - 0.009722B_2^3(t)$$

$$u^3(t) = 0.386364B_0^1(t) + 0.555303B_1^1(t) - 0.043750B_1^2(t) + 0.087500B_2^2(t)$$

For $n = 3$.

$$x^4(t) = 0.470833B_1^1(t) - 0.042235B_1^2(t) + 0.016516B_2^2(t)$$

$$-0.005520B_2^3(t) + 0.012606(t) - 0.003151B_3^4(t)$$

$$u^4(t) = 0.386364B_0^1(t) - 0.593111B_1^1(t) - 0.024842B_1^2(t)$$

$$+0.137922B_2^2(t) - 0.016807B_3^3(t)$$

Table 3 illustrates the present algorithms The results for this example in comparison with results obtained by Mehne [21]. further the optimal values for performance index J are also compared with the exact solution while the exact state and control solution as well as the actual value for J are Note that, the actual solution of this problem is:

$$x(t) = \frac{e(e^t - e^{-t})}{2(e^2 - 1)}, u(t) = \frac{e(e^t + e^{-t})}{2(e^2 - 1)} \text{ and}$$

$$J_{exact} = 0.30232588214$$

The comparison among the BEPs algorithm with $n = 1, 2, 3$ beside Mehne method [21] are listed in Table 3.

Table 3. *The values of cost functional J in example 4.*

Iteration	Our method	Error	Mehne method [21]	Error
1	0.3285984848	3.379×10^{-4}	0.333333333	5.0×10^{-3}
2	0.3267489571	2.1814×10^{-4}	0.3285984848	3.4×10^{-3}
3	0.3226487064	2.03089×10^{-4}	0.3284769571	2.1×10^{-4}

Note that the optimum values a_2, a_3, a_4 for minimizing the functional J are $a_2 = 0.113636$, $a_3 = 0.029167$, and $a_4 = 0.012606$.

5. Conclusion

A modification is proposed to the state parameterization technique by introducing accelerating iterative algorithm for solving optimal control problem with the aid of BEPs functions with only unknown coefficient must be evaluated in each approximation. A new resulted modification solution was constructed which based upon novel property of (BEPs) functions. The examples illustrated the reliability of the algorithm devoted in this paper.

References

[1] R. K. Pandey, N. Kumar, Solution of Lane–Emden type equations using Bernstein operational matrix of differentiation, New Astronomy. 17 (2012) 303–308.

[2] K. Rabiei, K. Parand, Collocation method to solve inequality-constrained optimal control problems of arbitrary order, Engineering with Computers. (2019) 1-11.

[3] S. Sabermahani, Y. Ordokhani, S.-A. Yousefi, Fractional-order Lagrange polynomials: An application for solving delay fractional optimal control problems, Transactions of the Institute of Measurement and Control. 41 (2019) 2997–3009.

[4] Y Edrisi-Tabri, M Lakestani, A Heydari, Two numerical methods for nonlinear constrained quadratic optimal control problems using linear B-spline functions, Iranian Journal of Numerical analysis and optimization, 6 (2) (2016) 17-37.

[5] B Kafash, A Delavarkhalafi, Numerical solution of nonlinear optimal control problems based on state parameterization, Iranian Journal of Science & Technology, 36 (3.1) (2012) 331-340.

[6] S. N Al-Rawi, F. A Al-Heety, S. S Hasan, A New Computational Method for Optimal Control Problem with B-spline Polynomials, Engineering and Technology Journal, 28 (18) (2010) 5711-5718

[7] S. N. Al-Rawi, H. R. Al-Rubaie, an Approximate solution of some continuous time Linear-Quadratic optimal control problem via Generalized Laguerre Polynomial, Journal of Pure and Applied Sciences, 22 (1) (2010) 85-97.

[8] GN. Elnagar, M. Razzaghi, A Chebyshev spectral method for the solution of nonlinear optimal control problems, Applied Mathematical Modeling, 21 (5) (1997) 255-260.

[9] J. Abed Eleiwy, S. N. SHIHAB, Chebyshev Polynomials and Spectral Method for Optimal Control Problem, Engineering and Technology Journal, 27 (14) (2009) 2642-2652.

[10] S. H. Mahdavi, H. Abdul Razak, An Efficient Iterative Scheme Using Family of Chebyshev's Operations, Mathematical Problems in Engineering. (2015) 1-10.

[11] S. N Shihab, Asmaa A Abdalrehman, Solving Optimal Control Linear Systems by Using New Third kind Chebyshev Wavelets Operational Matrix of Derivative, Baghdad Science Journal, Vol. 11, No. 2, pp. 229-234, (2014).

[12] B Kafash, A Delavarkhalafi, SM Karbassi, A numerical approach for solving optimal control problems using the Boubaker polynomials expansion scheme, Journal Interpolation and Approximation in Scientific Computing, (2014) 1-18.

[13] N. Ali, Numerical solution of 2D fractional optimal control problems by the spectral mwthod along with Bernstein operational matrix, International Journal of control, 91 (2018) 2632-2645.

[14] Y Edrisi Tabriz, A Heydari, Generalized B-spline functions method for solving optimal control problems, Computational Methods for Differential equations, 2 (4) (2014) 243-255.

[15] S. N Shihab, A. AA, M. N Mohammed Ali, Collocation Orthonormal Bernstein Polynomials Method for Solving Integral Equations, Engineering and Technology Journal, 33 (8) (2015)1502 -1493

[16] E. Safaie, M. H. Farahi, M. F. Ardehaie, An approximate method for numerically solving multi-dimensional delay fractional optimal control problems by Bernstein polynomials, Computational and Applied Mathematics, 34 (3) (2015) 831-846.

[17] M Alipour, R. A. Khan, H. Khan, K Karimi, Computational method Based on Bernstein polynomials for solving a fractional optimal control problem, Journal of Mathematics, 48 (1) (2016) 1-9.

[18] E. H. Doha, A. H. Bhrawy, M. A. Saker, Integrals of Bernstein polynomials: an application for the solution of high even-order differential equations, Applied Mathematics Letters, 24 (4) (2014) 559-565.

[19] A. Nemati, S. Yousefi, F. Soltania, An efficient numerical solution of fractional optimal control problems by using the Ritz method and Bernstein operational matrix, Asian Journal of Control, 18 (6) (2016) 2272-2282.

[20] S. Suman, A. Kumar, G. K. Singh, A new closed form method for design of variable bandwidth linear phase FIR filter using Bernstein multi-wavelets, International Journal of Electronics, 102 (4) (2015) 635-650.

[21] H. H. Mehne, A. H. Borzabadi, A numerical method for solving optimal control problems using state parametrization, Numer Algor, 42 (2006) 165-169.