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Homotopy Analysis Method for Nonlinear Fractional Gas Dynamics Equation

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Abstract

Fractional calculus is of vital importance and its significance is increased a lot since last many years. This paper applies Homotopy Analysis Method (HAM) to obtain analytical solutions of nonlinear fractional gas dynamics equation. Numerical results reveal the complete compatibility of proposed algorithm for such problems. Two special cases of the equation has been solved.

Keywords

Homotopy Analysis Method, Fractional Calculus, Nonlinear Fractional Gas Dynamics Equation

1. Introduction

Nowadays, noteworthy devotion in fractional differential equations has been put forth due to the immense demands in the areas of physics and engineering [1]. Many important phenomena in electromagnetic, viscoelasticity, acoustics and electrochemistry and material science are well described by differential equations of fractional order [2]. The homotopy analysis method (HAM) proposed by Shijun Liao (1992) is based on the concept of the homotopy. A homotopy describes a kind of continuous variation or deformation in mathematics. For example, a circle can be continuously deformed in to a square or an ellipse, the shape of a coffee cup can deform continuously into the shape of a doughnut. However, the shape of a coffee cup cannot be distorted continuously into the shape of a football.

Essentially, a homotopy defines a connection between different things in mathematics, which contain same characteristics in some aspects. In recent years, considerable interest in fractional differential equations has been stimulated due to their numerous applications in the areas of physics and engineering [3]. Mostly important phenomena in world are modeled and well described by fractional order differential equations [4-6]. Several excellent books and papers describing the state-of-the-art available in the literature testify to the maturity of theory of fractional order. Podlubny [7] provided the solutions method of differential equations of arbitrary real order and applications of the described methods in various fields. His book played an important role in the development of the theory of fractional order. The solution of differential equations of fractional order is much involved. Though many exact solutions for linear fractional differential equation had been found, in general, there exists no method that yields an exact solution for nonlinear fractional differential equations.

Recently, the semi-analytic techniques have been successfully employed to solve linear and nonlinear differential models, such as the differential transformation method [14–16], successive approximation method [17, 18], variational iteration method [19, 20], homotopy perturbation method [21] and other methods. The homotopy analysis method (HAM) [22–30] is one of the semi-analytical techniques used most often for solving various differential equations in fractional calculus. In this study, the homotopy analysis method is used to solve homogeneous nonlinear fractional gas dynamics equation

$$\frac{\partial^\alpha u}{\partial t^\alpha}+\frac{1}{2}(u^2)_x-u(1-u)=0, t>0, 0<\alpha\leq 1,$$

with initial condition

$$u(x,0)=g(x)$$

If $\alpha=1$, then the above Equation becomes classical gas dynamics equation.

2. Preliminaries and Basic Definitions

In this segment, we give some fundamental definitions and properties of the fractional calculus theory which will be used additional in this work. For the finite derivative in $[a,b]$ we define the following fractional integral and derivatives.

Definition 1. A real function $f(x), x>0$, is said to be in the space $C\mu, \mu\in R$, If there exists a real number $(p>\mu)$ such that $f(x)=x^p f_1(x)$, where $f_1(x)=C(0,\infty)$ and it is said to be in the space $C_\mu^m \mu$ if $f^m\in C\mu, m\in N$.

Definition 2. The Riemann-Liouville fractional integral operator of order $\alpha\geq 0$ of a function $f\in C\mu, \mu\geq -1$, is defined as

$$J^\alpha(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt, \alpha>0, x>0, J^0(x)=f(x).$$

Some properties of the operator J^a are discussed in the following

For $f\in C\mu, \mu\geq -1,, \alpha,\beta\geq 0$ and $\gamma\geq -1$

$$J^\alpha J^\beta f(x)=J^{\alpha+B}f(x),$$
$$J^\alpha J^B f(x)=J^B J^\alpha f(x),$$
$$J^\alpha x^\gamma=\frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}x^{\alpha+\gamma}.$$

The Riemann--Liouville derivative has convinced disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator suggested by M. Caputo in his struggle on the theory of viscoelasticity [8].

Definition 3. For m to be the smallest integer that exceeds, α the Caputo time fractional derivative operator of order $\alpha>0$ and defined as

$$D_t^\alpha f(x)=\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\begin{cases}\dfrac{1}{\Gamma(m-a)}\displaystyle\int_0^x (x-t)^{m-\alpha-1} f(t)dt, -1<<m, m\in N\\ \dfrac{\partial^\alpha u(x,t)}{\partial t^\alpha}, \alpha=m\end{cases}$$

Chain rule for fractional calculus and fractional complex transform:

In [9-12], the authors used the following chain rule $\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial u}{\partial s}\frac{\partial^\alpha s}{\partial t^\alpha}$ to convert a fractional differential equation with Jumarie's modification of Riemann-Liouville derivative into its classical differential part. In [13], the authors showed that this chain rule is invalid and show following relation.

$$D_t^a u=\sigma_t^{'}\frac{du}{d\eta}D_t^a\eta \text{ and } D_x^a u=\sigma_x^{'}\frac{du}{d\eta}D_x^a\eta,$$

To determine σ_s we consider a special case as follows $s=t^\alpha$ and $u=s^m$, we have

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\Gamma(1+m\alpha)t^{m\alpha-\alpha}}{\Gamma(1+m\alpha-\alpha)}=\sigma.\frac{\partial u}{\partial s}=\sigma m t^{m\alpha-\alpha}.$$

Thus we can calculate σ_s as

$$\sigma_s=\frac{\Gamma(1+m\alpha)}{\Gamma(1+m\alpha-\alpha)}.$$

Other fractional indexes $(\sigma_x^{'},\sigma_y^{'},\sigma_z^{'})$ can settle on in similar way. Li and He proposed fractional complex transform for converting fractional differential equations into ordinary differential equations, so that all analytical methods for advanced calculus can be easily applied to fractional calculus.

3. Homotopy Analysis Method (HAM)

We consider the following equation

$$\tilde{N}[u(\tau)]=0, \tag{1}$$

where $\tilde{N}$ is a nonlinear operator, τ denotes dependent variables and $u(\tau)$ is an unknown function. For simplicity, we ignore all boundary and initial conditions, which can be treated in the similar way. By means of HAM Liao constructed zero-order deformation equation

$$(1-p)\mathcal{L}[\emptyset(\tau;p)-u_0(\tau)]=p\hbar\tilde{N}[\emptyset(\tau;p)], \tag{2}$$

where $\mathcal{L}$ is a linear operator, $u_0(\tau)$ is an initial guess. $\hbar\neq 0$ is an auxiliary parameter and $p\in[0,1]$ is the embedding parameter. It is obvious that when p=0 and 1, it holds

$$\mathcal{L}[\emptyset(\tau;0)-u_0(\tau)]=0 \implies \emptyset(\tau;0)=u_0(\tau), \tag{3}$$

$$\hbar\tilde{N}[\emptyset(\tau;1)]=0 \implies \emptyset(\tau;1)=u(\tau), \tag{4}$$

respectively. The solution $\emptyset(\tau;p)$ varies from initial guess $u_0(\tau)$ to solution $u(\tau)$. Liao expanded $\emptyset(\tau;p)$ in Taylor series about the embedding parameter

$$\emptyset(\tau;p) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau)p^m, \tag{5}$$

where

$$u_m(\tau) = \frac{1}{m!}\frac{\partial^m \emptyset(\tau;p)}{\partial p^m}\Big|_{p=0} \tag{6}$$

The convergence of (5) depends on the auxiliary parameter $\hbar$. If this series is convergent at p=1, one has

$$\emptyset(\tau;1) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau), \tag{7}$$

Define vector

$$\vec{u}_n = \{u_0(\tau), u_1(\tau), u_2(\tau), u_3(\tau), \dots\dots\dots, u_n(\tau)\}$$

If we differentiate the zeroth-order deformation equation Eq. (2) m-times with respect to p and then divide them $m!$ and finally set $p = 0$, we obtain the following mth-order deformation equation

$$\mathcal{L}[u_m(\tau) - \chi_m u_{m-1}(\tau)] = \hbar\Re_m(\vec{u}_{m-1}), \tag{8}$$

Where

$$\Re_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\tilde{N}[\emptyset(\tau;p)]}{\partial p^{m-1}}\Big|_{p=0} \tag{9}$$

and

$$\chi_m = \begin{cases} 0, m \le 1, \\ 1, m > 1, \end{cases} \tag{10}$$

If we multiply with $\mathcal{L}^{-1}$ each side of Eq. (8), we will obtain the following mth order deformation equation

$$u_m(\tau) = \chi_m u_{m-1}(\tau) + \hbar\, \Re_m\, (\vec{u}_{m-1})$$

4. Solution Procedure

In this section, we solve two problems of nonlinear fractional gas dynamics equation to demonstrate the efficiency of the Homotopy Analysis Method.

Problem 1: Consider the following gas dynamics equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{1}{2}(u^2)_x - u(1-u) = 0, 0 < \alpha \le 1,$$

with initial condition

$$u(x,0) = e^{-x}.$$

Now we apply the HAM to solve gas dynamics equation. The solution $u(x,t)$ can be expressed by a set of base functions.

$$\{t^n | n = 0,1,2, \dots\dots\dots..\},$$

In the following forms

$$u(x,t) = \sum_{n=o}^{+\infty} a_n t^n,$$

Where a_n is coefficient. This provides us with the first rule of solution expression. Under the rule of solution expression and according to initial condition, it is straightforward to choose

$$u_0(x,t) = e^{-x},$$

As the initial approximations of $u(x,t)$ to choose the auxiliary linear operator

$$L[\phi(t;q)] = \frac{\partial^\alpha \phi(t;q)}{\partial t^\alpha},$$

With the property

$$L[C] = 0,$$

Where C is an integral constant. Furthermore, we define a system of nonlinear operators as

$$N[\phi(t;q)] = \frac{\partial^\alpha \phi(t;q)}{\partial t^\alpha} - \frac{\partial^2 \phi}{\partial x^2} + \phi(t;q),$$

Using the above definition, we construct the zeroth-order deformation equation

$$(1-q)\mathcal{L}[\phi(t;q) - u_0(x,t)] = q\hbar N[\phi(t;q)],$$

Obviously, when q=0 and q=1,

$$\phi(t;0) = u_0(x,t), \phi(t;1) = u(x,t),$$

Therefore as the embedding parameter q increases from 0 to 1, The solution $\emptyset(t;q)$ varies from the initial guess to the solution for i=1, 2. Expanding $\emptyset(t;q)$ in Taylor series with respect to q, one has:

$$\emptyset(t;q) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t)q^m,$$

Where

$$u_m(t) = \frac{1}{m!}\frac{\partial^m \emptyset(t;q)}{\partial q^m}|_{q=0}$$

Define the vector

$$\bar{u}_n = \{u_0(t), u_1(t), \dots\dots\dots., u_n(t)\}.$$

Differentiating the zero-order deformation equation m-times with respect to q, and finally dividing by m!,

We gain the mth order deformation equations

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar R_m(\bar{u}_{m-1}),$$

Subject to initial condition $u_m(x,0) = 0,$

$$R_m(\bar{u}_{m-1}) = \frac{\partial^\alpha u_{m-1}}{\partial t^\alpha} + \frac{1}{2}(u_{m-1}^2)_x - u_{m-1}(1-u_{m-1}),$$

Now the solution of the mth-order deformation equation for m ≥1 becomes

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar j_t^\alpha[R_m(\bar{u}_{m-1})].$$

We now successfully obtain

$$u_1 = -\hbar e^{-x}\frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$u_2 = -\hbar e^{-x}\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \hbar^2 e^{-x}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \hbar^2 e^{-x}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$\vdots$$

Then the solution expression can be written in the form

$$u(x,t) = \sum_{n=o}^{+\infty} a_n t^n ,$$

$$u(x,t) = e^{-x} - 2\hbar e^{-x}\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \hbar^2 e^{-x}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \hbar^2 e^{-x}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots.$$

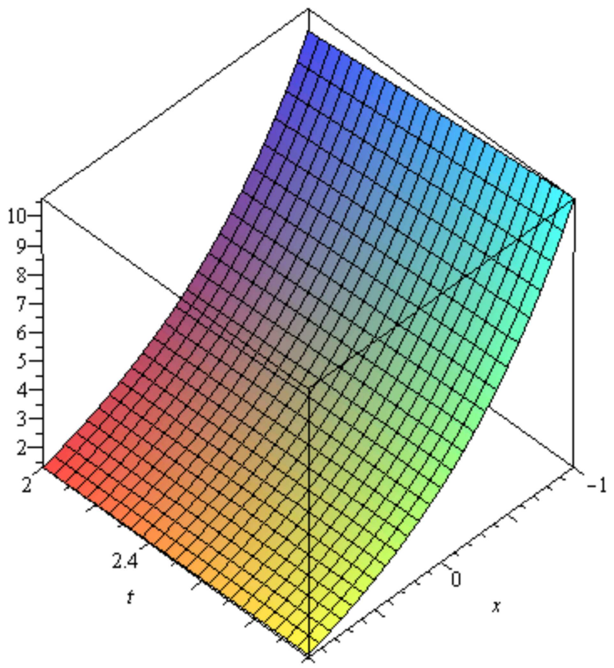

Figure 1. *Solution for α=0.25.*

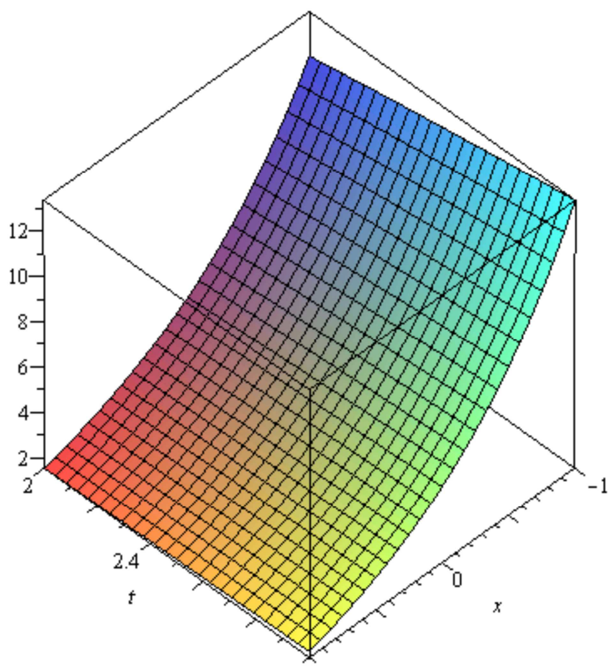

Figure 2. *Solution for α=0.50.*

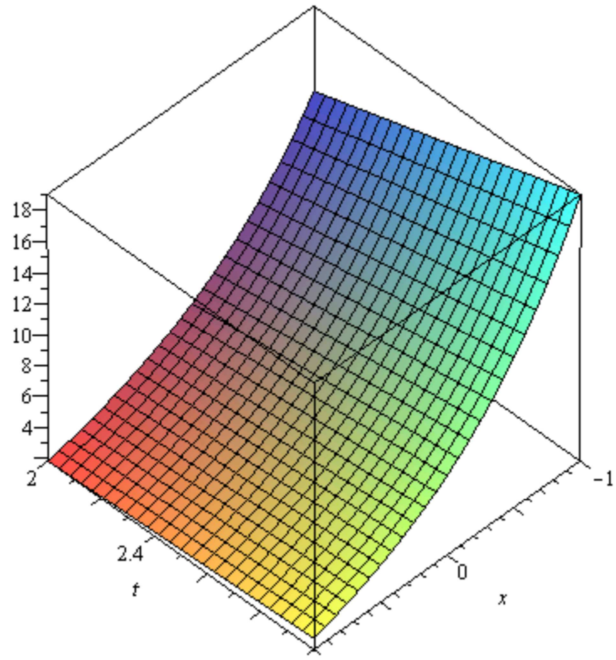

Figure 3. *Solution for α=1.*

Problem 2: Consider the following gas dynamics equation

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} + \frac{1}{2}(u^2)_x + (1+t)^2 u^2 = x^2, 0 < \alpha \le 1,$$

with initial condition

$$u(x,0) = x.$$

Now we apply the HAM to solve gas dynamics equation. The solution $u(x,t)$ can be expressed by a set of base functions.

$$\{t^n | n = 0,1,2, \ldots \ldots \ldots ..\},$$

In the following forms

$$u(x,t) = \sum_{n=o}^{+\infty} a_n t^n ,$$

Where a_n is a coefficient. This provides us with the first rule of solution expression. Under the rule of solution expression and according to initial condition, it is straightforward to choose

$$u_0(x,t) = x,$$

As the initial approximations of $u(x,t)$ to choose the auxiliary linear operator

$$L[\phi(t;q)] = \frac{\partial^{\alpha}\phi(t;q)}{\partial t^{\alpha}},$$

With the property

$$L[C] = 0,$$

Where C is a integral constant. Furthermore, we define a system of nonlinear operators as

$$N[\phi(t;q)] = \frac{\partial^{\alpha}\phi(t;q)}{\partial t^{\alpha}} - \frac{\partial^2 \phi}{\partial x^2} + \phi(t;q),$$

Using the above definition, we construct the zeroth-order deformation equation

$$(1-q)\mathcal{L}[\phi(t;q)-u_0(x,t)] = q\hbar N[\phi(t;q)],$$

Obviously, when q=0 and q=1,

$$\phi(t;0) = u_0(x,t), \phi(t;1) = u(x,t),$$

Therefore as the embedding parameter q increases from 0 to 1, The solution $\emptyset(t;q)$ varies from the initial guess to the solution for i=1,2. Expanding $\emptyset(t;q)$ in Taylor series with respect to q, one has:

$$\emptyset(t;q) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t)q^m,$$

Where

$$u_m(t) = \frac{1}{m!}\frac{\partial^m \emptyset(t;q)}{\partial q^m}|_{q=0}$$

Define the vector

$$\bar{u}_n = \{u_0(t), u_1(t), \dots \dots \dots, u_n(t)\}.$$

Differentiating the zero-order deformation equation m-times with respect to q, and finally dividing by m!,

We gain the mth order deformation equations

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar R_m(\bar{u}_{m-1}),$$

Subject to initial condition $u_m(x,0) = 0$,

$$R_m(\bar{u}_{m-1}) = \frac{\partial^\alpha u_{m-1}}{\partial t^\alpha} - \frac{1}{2}(u_{m-1}^2)_x + (1+t)^2 u_{m-1}^2 - x^2,$$

Now the solution of the mth-order deformation equation for m ≥1 becomes

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + h j_t^\alpha[R_m(\bar{u}_{m-1})].$$

We now successfully obtain

$$u_1 = \hbar\left[x\frac{t^\alpha}{\Gamma(\alpha+1)} + 2x^2\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + 2x^2\frac{t^{\alpha+2}}{\Gamma(\alpha+3)}\right],$$

$$\vdots$$

Then the solution expression can be written in the form

$$u(x,t) = \sum_{n=o}^{+\infty} a_n t^n,$$

$$u(x,t) = x + \hbar\left[x\frac{t^\alpha}{\Gamma(\alpha+1)} + 2x^2\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + 2x^2\frac{t^{\alpha+2}}{\Gamma(\alpha+3)}\right] + \cdots.$$

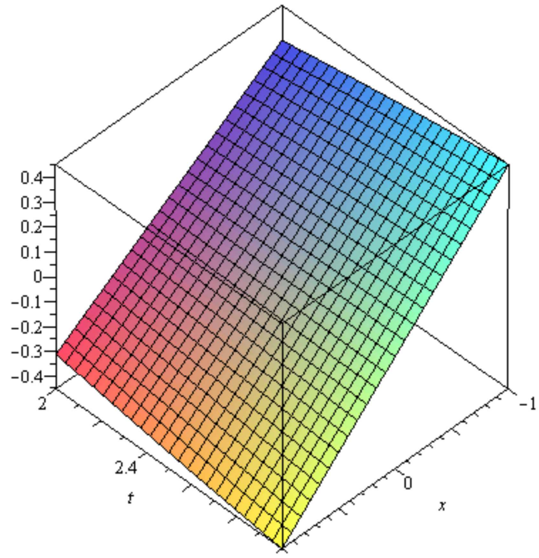

Figure 4. *Solution forα=0.25.*

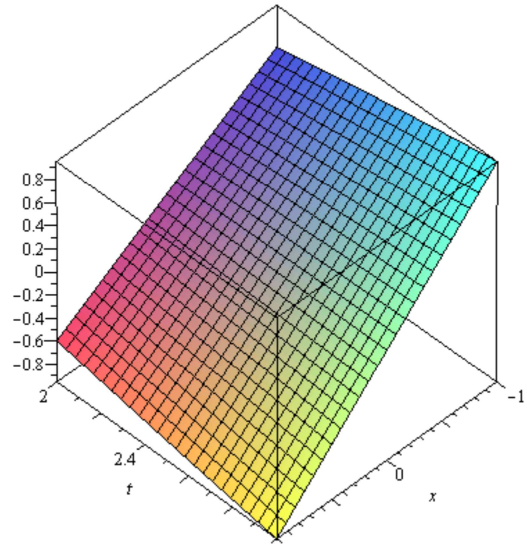

Figure 5. *Solution for α=0.50.*

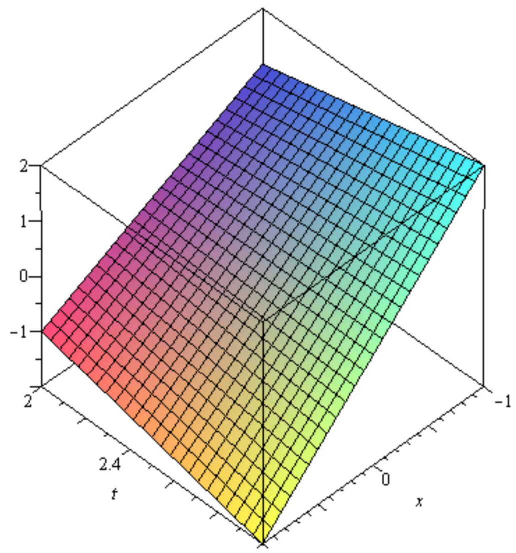

Figure 6. *Solution for α=1.*

5. Conclusion

This article is devoted to attain, test and analyze the novel solutions and physical properties of nonlinear fractional order gas dynamics equations. For this, fractional order nonlinear differential equation is considered and we apply homotopy Analysis Method (HAM). We attain desired analytic solutions of various types for different values of parameters. It is guaranteed the accuracy of the attain results by backward substitution into the original equation with Maple 16. The scheming procedure of this method is simplest, straight and productive. We observed that the under study technique is more reliable and have minimum computational task, so widely applicable. In precise we can say this method is quite competent and much operative for evaluating exact solution of NLEEs. The validity of given algorithm is totally hold up with the help of the computational work, the graphical representations and successive results. Results obtained by this method are very encouraging and reliable for solving any other type of NLEEs. The graphical representations clearly indicate the solitary solutions.

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