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On Fractional Governing Equations of Spherical Particles Settling in Water

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Abstract

This paper shows a structure to get the result to the uneven settle actions of few solid spherical particles declining in water as a Newtonian fluid by homotopy analysis method. The partial derivative is described in Modified Riemann liouville sense. This method performs very well in competence. Numerical results explain the whole consistency in used algorithm.

Keywords

Homotopy Analysis Method, Spherical Particles, Drag Coefficient, Fractional Calculus, Sedimentation Phenomenon, Modified Riemann-Liouville Fractional Derivative

1. Introduction

In current time, the fractional order differential equations have been happening in many substantial and engineering problems such like frequency dependent damp activities of material, diffusion processes, motion of a large thin plate in a Newtonian fluid, creeping and relaxation functions for viscoelastic materials. For more details on the applications of fractional derivatives in variety and statistical mechanics see [1-4]. Most fractional differential equations do not have accurate analytical solutions, therefore approximate and numerical techniques must be used. Learning of engrossed bodies motion in fluids has long been a subject of great interest due to its massive applications in nature and industry e.g. Sediment transport and deposition in pipelines. The settling of an entity, including a solid particle, bubble, or drop, both in a Newtonian fluid and in a non-Newtonian fluid, is reported by Bridge and Bennett [5] and Chhabra [6]. Haider and Levenspiel [7] offered several heave coefficients for spherical and non-spherical particles [8].

A particle falling vertically in a fluid under the influence of gravity will accelerate until the gravitational force is reasonable by the struggle forces, including buoyancy and drag forces. When the particle reaches to a constant velocity, it's called as "terminal velocity" or "settling velocity". The familiarity of the terminal velocity of solids declining in liquids is required in many industrial applications such as mineral processing, solid-liquid mixing, hydraulic transport, slurry systems, rasping water jets, fluidized bed reactors and so on. It is unambiguous that most of the pervious investigations are carried out for steady-state conditions, where the particles attain to terminal velocity, and slight of them has been reported about the unsteady motion of spherical objects.

2. Mathematical Formulation

For modeling the particle sediment phenomenon, consider a small, rigid spherical, non-deformable shape of diameter D, mass m and density ρ_s as particle which is falling in infinite extent filled water as an incompressible Newtonian fluid. Density of water ρ and its viscosity μ are known. We just considered the gravity, buoyancy and drag forces on particle and assumed $\rho << \rho_s$.

Rewriting force balance for particle, the equation of motion is as follows

$$m\frac{dw}{dt}=mg\left(1-\frac{\rho}{\rho_s}\right)-\frac{1}{8}\pi D^2\rho C_D w^2-\frac{1}{12}\pi D^3\rho w, \quad (1)$$

where C_D is the drag coefficient, in the right hand side of the Eq. (1), the first term represents the buoyancy affect, the second term corresponds to drag resistance, and the last term is due to the added mass effect which is due to acceleration of fluid around the particle. The main difficulty to solve Eq. (1) is non-linear terms due to the non-linearity nature of the drag coefficient C_D Ferreira et al. [9], in their analytical study, suggested a correlation for C_D of spherical particles which has good agreement with the experimental data in a wide range of Reynolds number, $0\le Re\le 10^5$ and C_D is given by

$$C_D=\frac{24}{Re}\left(1+\frac{1}{48}Re\right) \quad (2)$$

The mass of the spherical particle is

$$m=\frac{1}{6}\pi D^3\rho_s \quad (3)$$

Substituting Equations (2) and (3) into Eq. (1), we have

$$a\frac{dw}{dt}+bw+cw^2-d=0,\ w(0)=0 \quad (4)$$

where

$$a=\frac{1}{12}\pi D^3(2\rho_s+\rho) \quad (5)$$

$$b=3\pi D_\mu \quad (6)$$

$$c=\frac{1}{16}\pi D^2\rho \quad (7)$$

$$d=\frac{1}{6}\pi D^3 g(\rho_s-\rho) \quad (8)$$

In recent years there has been a great deal of interest in fractional differential equations. These equations arise in continuous time random walks, modeling of anomalous diffusive and sub diffusive systems, unification of diffusion and wave propagation phenomenon, and simplification of the results and more applications were studied in [10, 11].

Our concern in this work is to consider the analytical solution of the nonlinear differential equation with time-fractional derivatives of the form:

$$a\frac{d^\beta w}{d\,t^\beta}+bw+cw^2-d=0\ w(0)=0\ 0<\beta\le 1, t> \quad (9)$$

Equation (9) reduces to the classical nonlinear differential equation (4) for $\beta=1$. The objective of this paper is to extend the application of the homotopy analysis method (HAM) by using modified Reimann-Liouville derivative [12-16] to obtain analytic solutions to the time-fractional equation of some spherical particles settling in water. The homotopy analysis method is a computational method that yields analytical solutions and has certain advantages over standard numerical methods. It is free from rounding off errors, as it does not involve discretization, and does not require large computer obtained memory or power. The method introduces the solution in the form of a convergent fractional series with elegantly computable terms.

The HAM is developed in 1992 by Liao in [17-26]. By the present method, numerical results can be obtained with using a few iterations. The HAM contains the auxiliary parameter $\hbar$, which provides us with a simple way to adjust and control the convergence region of solution series for large values of t. Unlike, other numerical methods are given low degree of accuracy for large values of *t*. Therefore, the HAM handles linear and nonlinear problems without any assumption and restriction.

3. Modified Riemann-Liouville Derivative

Assume $h:R\to R, x\to h(x)$ denote a continuous (but not necessarily differentiable) function and let the partition $h>0$ in the interval [0,1]. Through the fractional Riemann Liouville integral

$${}_0I_x^\beta h(x)=\frac{1}{\Gamma\alpha}\int_0^x(x-\psi)^{\beta-1}f(\psi)d\psi,\quad \beta>0 \quad (10)$$

The modified Riemann-Liouville derivative is defined as

$${}_0D_x^\beta h(x)=\frac{1}{\Gamma(n-\beta)}\frac{d^n}{dx^n}\int_0^x(x-\psi)^{n-\beta}(f(\psi)-f(0))\,d\psi, \quad (11)$$

Where $x\in[0,1]$, $n-1<\beta\le n$ and $n\ge 1$

G. Jumarie's derivative is defined through the fractional difference

$$\Delta^\beta=(FW-1)^\beta h(x)=\sum_0^\infty(-1)^k\binom{\beta}{k}f[x+(\beta-k)h], \quad (12)$$

Where $FW\,h(x)=h(x+h)$. Then the fractional derivative is defined as the following limit,

$$f^{(\beta)}=\lim_{h\to 0}\frac{\Delta^\beta f(x)}{h^\beta} \quad (13)$$

The proposed modified Riemann–Liouville derivative as shown in equation (11) is strictly equivalent to equation. (13). Meanwhile, we would introduce some properties of the fractional modified Riemann–Liouville derivative in equations. (14) and (15).

(i) Fractional Leibniz product law

$$ {}_0D_x^{\beta}(wv) = w^{(\beta)}v + wv^{(\beta)} \tag{14}$$

(ii) Fractional Leibniz formulation

$$ {}_0I_x^{\beta}\,{}_0D_x^{\beta}h(x) = h(x) - h(0), \quad 0 < \beta \le 1 \tag{15}$$

Therefore, the integration by part can be used during the fractional calculus

$$ {}_{\beta}I_{b_1}^{\beta}w^{(\beta)}v = (wv)/_{a_1}^{b_1} - {}_{a_1}I_{b_1}^{a_1}wv^{(\beta)} \tag{16}$$

(iii) Integration with respect to $(d\psi)^{\beta}$.

Assume $h(x)$ denote a continuous $R \to R$ function, we use the following equality for the integral with respect to $(dw)^{\alpha}$

$$ \begin{aligned} I_x^{\beta}h(x) &= \frac{1}{\Gamma\mathrm{B}}\int_0^x (x-\psi)^{\beta-1}f(\psi)\,d\psi, \quad 0<\beta\le 1 \\ &= \frac{1}{\Gamma(1+\beta)}\int_0^x f(\psi)d(\psi)^{\beta} \end{aligned} \tag{17}$$

4. Homotopy Analysis Method (HAM)

We consider the following differential equation

$$ HD\left[w(x,t)\right] = 0, \tag{18}$$

Where HD *is* a nonlinear operator for this problem, x and t denote an independent variables, $w(x,t)$ is an unknown function.

In the frame of HAM, we can construct the following zeroth-order deformation:

$$ (1-q)L\big(w(x,t;q) - w_0(x,t)\big) = q\hbar H(x,t)HD\big(w(x,t;q)\big), \tag{19}$$

where $q \in [0, 1]$ is the embedding parameter, $\hbar \ne 0$ is an auxiliary parameter, $H(x,t) \ne 0$ is an auxiliary function, L is an auxiliary linear operator, $w_0(x,t)$ is an initial guess of $w(x,t)$ and $w(x,t;q)$ is an unknown function of the independent variables *x, t* and *q*.

Obviously, when $q = 0$ and $q = 1$, it holds

$$ w(x,t;0) = w_0(x,t),\ w(x,t;1) = w(x,t), \tag{20}$$

Using the parameter q, we expand $W(x,t;q)$ in Taylor series as follows:

$$ w(x,t;q) = w_0(x,t) + \sum_{r=1}^{\infty} w_r(x,t)q^r, \tag{21}$$

where

$$ w_r = \frac{1}{r!}\frac{\partial^r w(t;q)}{\partial^r q}\bigg|_{q=0} \tag{22}$$

Assume that the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$ and the auxiliary function $H(x,t)$ are selected such that the series (19) is convergent at $q = 1$, then due to (20) we have

$$ w(x,t) = w_0(x,t) + \sum_{r=1}^{\infty} w_r\ (x,t) \tag{23}$$

Let us define the vector

$$ w_n(x,t) = \{w_0(x,t), w_1(x,t), \ldots, w_n(x,t)\} \tag{24}$$

Differentiating (11) r times with respect to the embedding parameter q, then setting $q = 0$ and finally dividing them by $r!$, we have the so-called rth-order deformation equation

$$ L\left[W_r(x,t) - \chi_r w_{r-1}(x,t)\right] = \hbar H(x,t)R_r(w_{r-1}), \tag{25}$$

where

$$ R_r(w_{r-1}) = \frac{1}{(r-1)!}\frac{\partial^{r-1}HD\big(w(t;q)\big)}{\partial^{r-1}q}\bigg|_{q=0}, \tag{26}$$

$$ \text{and } \chi_r = \begin{cases} 0 & r \le 1, \\ 1 & r > 1. \end{cases} \tag{27}$$

Finally, for the purpose of computation, we will approximate the HAM solution (23) by the following truncated series:

$$ \varphi_r(t) = \sum_{k=0}^{r-1} w_k(t). \tag{28}$$

5. Applications

In this section, we demonstrate the efficiency and effectiveness of the Homotopy analysis method with modified Riemann–Liouville derivative.

For the case, $a = b = c = d = 1$, eq. (9) becomes

$$ \frac{d^{\beta}w(t)}{dt^{\beta}} + w(t) + w^2(t) - 1 = 0, \quad 0 < \beta \le 1, \tag{29}$$

Subject to the initial condition

$$ w(0) = 0.$$

Constructing the following Homotopy, According to (19), the zeroth-order deformation can be given by

$$(1-q)L\left(w(x,t;q)-w_0(x,t)\right)$$
$$=q\hbar H(x,t)\left(D_\beta^t w(x,t;q)+w(x,t;q)+w^2(x,t;q)-1\right)$$

We can start with an initial approximation $w_0(x,t)=0$ and we choose the auxiliary linear operator

$$L\left(w(x,t;q)\right)=D_\beta^t w(x,t;q),$$

with the property

$$L(C)=0,$$

where C is an integral constant. We also choose the auxiliary function to be

$$H(x,t)=1.$$

Hence, the rth-order deformation can be given by

$$L\left[w_r(x,t)-\chi_r w_{r-1}(x,t)\right]=\hbar H(x,t)R_r(uw_{r-1}),$$

where

$$R_r(w_{r-1})=D_\beta^t(w_{r-1})+w_{r-1}+\sum_{i=0}^{r-1}w_i w_{r-1-j}-1 \tag{30}$$

Now the solution of the rth-order deformation equations (24) for $r\geq 1$ become

$$w_r(x,t)=\chi_r w_{r-1}(x,t)+\hbar L^{-1}\left[R_r(w_{r-1})\right]. \tag{31}$$

Consequently, (for $\hbar=-1$) the first few terms of the HAM series solution are as follows:

$$w_0(x,t)=0,$$

$$w_1(x,t)=\frac{t^\beta}{\Gamma(1+\beta)},$$

$$w_2(x,t)=-\frac{t^{2\beta}}{\Gamma(1+2\beta)},$$

$$w_3(x,t)=\left(\frac{1}{\Gamma(1+3\beta)}-\frac{\Gamma(1+2\beta)}{\Gamma^2(1+\beta)\Gamma(1+3\beta)}\right)t^{3\beta},$$

$$w_4(x,t)=\left(-\frac{1}{\Gamma(1+4\beta)}+\frac{\Gamma(1+2\beta)}{\Gamma^2(1+\beta)\Gamma(1+4\beta)}+\frac{2\Gamma(1+3\beta)}{\Gamma(1+4\beta)\Gamma(1+2\beta)\Gamma(1+\beta)}\right)t^{4\alpha},$$

and so on. Hence, the HAM series solution (for $\hbar=-1$) is

$$w(x,t)=w_0(x,t)+w_1(x,t)+w_2(x,t)+..$$

$$w(t)=\frac{t^\beta}{\Gamma(1+\beta)}-\frac{t^{2\beta}}{\Gamma(1+2\beta)}+\left(\frac{1}{\Gamma(1+3\beta)}-\frac{\Gamma(1+2\beta)}{\Gamma^2(1+\beta)\Gamma(1+3\beta)}\right)t^{3\beta}$$
$$+\left(-\frac{1}{\Gamma(1+4\beta)}+\frac{\Gamma(1+2\beta)}{\Gamma^2(1+\beta)\Gamma(1+4\beta)}+\frac{2\Gamma(1+3\beta)}{\Gamma(1+4\beta)\Gamma(1+2\beta)\Gamma(1+\beta)}\right)t^{4\beta}+..., \tag{32}$$

For $\beta=1$, the equation (31) can be reduced as

$$w(t)=t-\frac{1}{2}t^2-\frac{1}{6}t^3+\frac{7}{24}t^4-\frac{1}{24}t^5... \tag{33}$$

6. Conclusion

In given paper, we use HAM to get the solutions of the Equation of some spherical particles settling in water. The HAM is straightforward without restrictive assumptions, and the components of the series solution can be easily computed using any mathematical symbolic package. The paper presents that homotopy analysis method can easily be used to construct solutions for a broad class of nonlinear problems with fractional derivatives.

Nomenclature

a_1, b_1, c_1, d_1	Constants	Acc	Acceleration [m/s^2]
t	Time [s]	w	Velocity [m/s]
C_D	Drag coefficient	D	Particle diameter [m]
g	Acc due to gravity [m/s^2]	m	Particle mass [kg]
μ	Dynamic viscosity [$\frac{kg}{ms}$]	ρ	Fluid density [$\frac{kg}{m^3}$]

ρ_s Spherical particle density [kg/m^3]

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