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Cardan’s Coupling Shaft as a Dynamic Evolutionary System

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Abstract

The authors have long been concerned with the problem of dimensioning the long articulated shafts of trucks. The impulse for launching the research was a failure on real vehicles - permanent bending deformation of the Cardan coupling shafts. At present, this issue is solved in the framework of grant TA04010579 of the Czech Republic’s Technology Agency focusing on gear pumps. Regarding the shaft system with Hooke’s joints, it is necessary to respect the relative transverse oscillation of the shaft, which introduces to the system the frequency spectrum dependence on the angular velocity of the rotation. The paper describes the mathematical model derivation of the problem. The proposed model is extensively solved using Transfer-matrix Method and the Finite Element Method, and successfully used to detect the causes of failures that occurred on real vehicles.

Keywords

Transverse Vibration, Articulated Shafts, Frequency Spectrum, Rotational Resonance

1. Introduction

Propeller shafts of drive vehicles transmit a torque at relatively large distances. The shafts are based on long and slender, and must be dimensioned not only in terms of torsional stress, but it is also necessary to monitor its resistance to lateral vibration. Due to the continuous operational area, the shafts are needed to operate in subcritical speed. Results of previous works which were also confronted with experiments showed that the propeller shafts represent strong evolutionary systems (increasing the angular velocity of rotation significantly reduce the spectrum of natural frequency relative lateral vibrations) and in practical calculations it is necessary to respect this influence. For that reason, it is not possible to model the shafts using procedures that are commonly reported in the literature, but it is necessary to formulate a model that allows this effect respected.

2. Formulation of a Problem, Used Processing Methods

Propeller shafts are in a steady state stressed by excitation bending moments harmonic, and their vectors are orthogonal to the rotating plane of a relevant fork Hook´s joints (Figure 1). The drive torque mentioned generate in a steady state due to the transmission flow through Hook´s joints and cause lateral oscillations of the propeller shafts in its rotating space. In formulating a mathematical model, it is necessary to start from the assumption of formation relative spatial bending vibration in the shaft system 0(x, y, z) (Figure 2), which rotates at an angular speed $\vec{\dot\varphi}_x$. If we neglect the Coriolis force and gyroscopic moments acting on the element of the shaft, we can solve the problem in the rotating plane 0(x, y). The instantaneous state of the element is determined by the angular velocity $\vec{\dot\varphi}_x$, the velocity

$\vec{v}$ and the angular velocity $\vec{\dot{\varphi}}_{z_s}$. This article aims to build a mathematical model of a coupling shaft to calculate spectral and modal properties of the connecting shaft with respect to the field of centrifugal forces that is causing the addiction of natural frequencies of bending vibrations relative to the angular velocity of the shaft´s rotation.

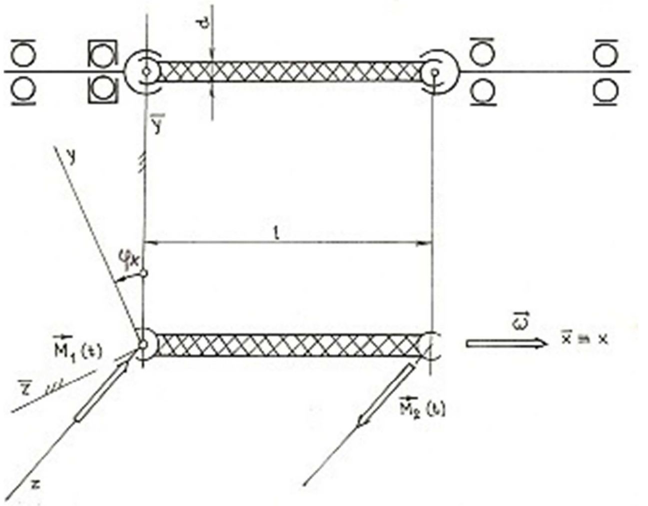

Figure 1. *Cardan coupling shaft as a one-dimensional linear continuum.*

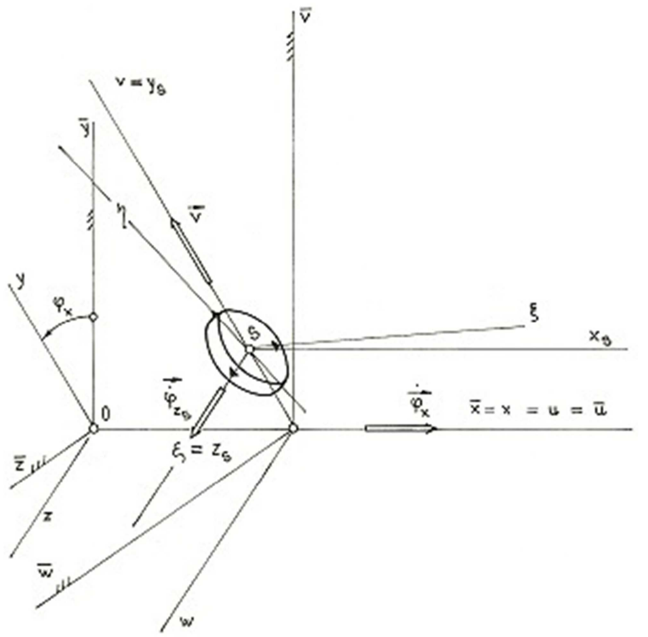

Figure 2. *Element of a one-dimensional linear continuum in a state of combined bending circular oscillation.*

The problem is solved step by step. In the third chapter, there is constructed by the method of physical discretization a simple model (of the solved problem), which is evident from the nature of the centrifugal force field´s influence on the spectral properties of the shaft. In the chapter 4, there is performed an analytical solution of speed resonances propshaft's test model, whose aim is to obtain values for verification subsequently processed models based on the transfer-matrix method and the finite element method.

3. Solving the Problem by Physical Discretization

Let´s replace the driveshaft which is shown in the Figure 1 (consider solid bearings) by discrete mechanical system with one degree of freedom. Divide it into two equal halves which will represent an intangible spring (Figure 3) having rigidity $\frac{k}{2}$.

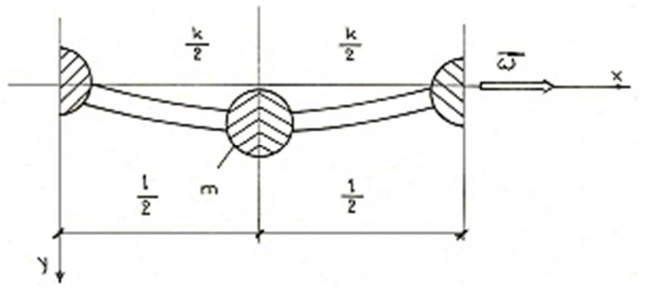

Figure 3. *Schematic diagram of the physical displacement of the articulated shaft.*

Concentrate the weight to endpoints of the springs so that two fixed points belong to their supports (for x = 0, resp. x = 1), the other two fixed points merge into one with a weight m = Slρ /2 in the middle of the shaft (x = 1/2). The resulting discrete model of a connecting shaft is shown in the Figure 4. Let´s determine the spring stiffness of the relation:

$$k = \frac{48EJ}{l^3}, \tag{1}$$

further determine weight discrete compensation (weight of fixed point) from the relationship

$$m = \frac{\rho Sl}{2}, \tag{2}$$

where

$$J = \frac{\pi}{4} r^4,\ S = \pi r^2. \tag{3}$$

Description of physical quantities above relationships is the following:

l [m]... shaft length,

r [m]... radius shaft,

ρ [kgm^{-3}]... material density,

E [Pa]... modulus of elasticity in tension.

Assuming a constant angular velocity $\overline{\omega}$, it is necessary to induce the moment $\vec{M}$. We compile equations of motion using the Lagrange´s equations. For generalized coordinates we choose rotation φ_x and displacement y (in the static equilibrum position, there assumed fixed point on the axis of rotation). The kinetic energy of a fixed point is expressed in the form:

$$E_k = \frac{1}{2} m\dot{y}^2 + \frac{1}{2} m (y\dot{\varphi}_x)^2, \tag{4}$$

the potential energy stored in the spring is expressed in the form:

$$V = \frac{1}{2} ky^2. \tag{5}$$

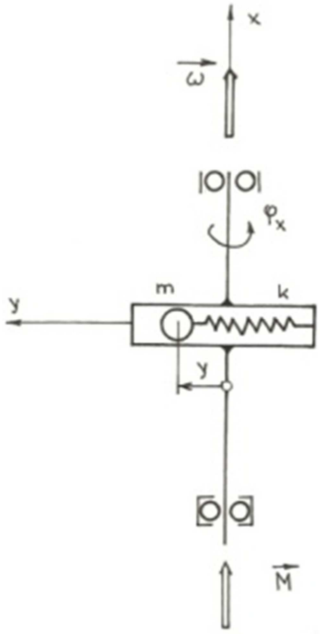

Figure 4. *Discrete model of the continuum section in the state of relative transverse oscillations*

We receive equations of motion in the form:

$$m\ddot{y} + (k - m\omega^2)\, y = 0,\ M - 2my\dot{y}\omega = 0 \tag{6}$$

Then we rewrite the equation of relative oscillating movement in the rotating plane in the form:

$$\ddot{y} + \Omega^2 y = 0, \tag{7}$$

where

$$\Omega = \sqrt{\frac{k}{m} - \omega^2} \tag{8}$$

it is the natural frequency of relative undamped oscillations.

From the equation (8) it is obvious that the frequency of relative oscillations depends on the relative angular velocity of rotation. The system is obvious stable only in cases the following condition is satisfied:

$$\omega^2 < \frac{k}{m}. \tag{9}$$

We modify the equation (8) into this form:

$$\Omega^2 + \omega^2 = \frac{k}{m}, \tag{10}$$

which is the equation of a circle with a radius $\sqrt{\frac{k}{m}}$ centered at the origin of the system $0(\omega, \boldsymbol{\Omega})$. Due to the physical nature of the problem, it applies to only a quarter of the circle in the first quadrant. The above procedure will be applied to the propeller shaft of the following parameters (it is the propeller shaft of the vehicle Š 781):

$$r = 0{,}0105\ [m],\ l = 0{,}65\ [m],$$

$$E = 2{,}1.10^{11}\ [Pa],\ \rho = 7{,}8.10^3 [kg.m^{-3}].$$

You will receive the following calculations:

$$J = \frac{\pi}{4} r^4 = \frac{\pi}{4} (0{,}0105)^4 = 9 \cdot 10^{-4}\ [m^4],$$

$$S = \pi r^2 = \pi\ (0{,}0105)^2 = 3{,}46 \cdot 10^{-4}\ [m^2],$$

$$k = \frac{48EJ}{l^3} = \frac{48 \cdot 2{,}1 \cdot 10^4 \cdot 9 \cdot 10^{-4}}{0{,}65^3} = 3{,}3 \cdot 10^5 [N/m],$$

$$m = \frac{\rho S l}{2} = \frac{7{,}8 \cdot 10^3 \cdot 3{,}46 \cdot 10^{-4} \cdot 0{,}65}{2} = 0{,}88\ [kg],$$

$$\boldsymbol{\Omega}\ (0) = \left(\frac{k}{m}\right)^{\frac{1}{2}} = \left(\frac{3{,}3 \cdot 10^5}{0{,}88}\right)^{\frac{1}{2}} = 612{,}4\ [rads^{-1}].$$

In this case, the natural frequency dependence of the relative transversal oscillations is shown in the Figure 5.

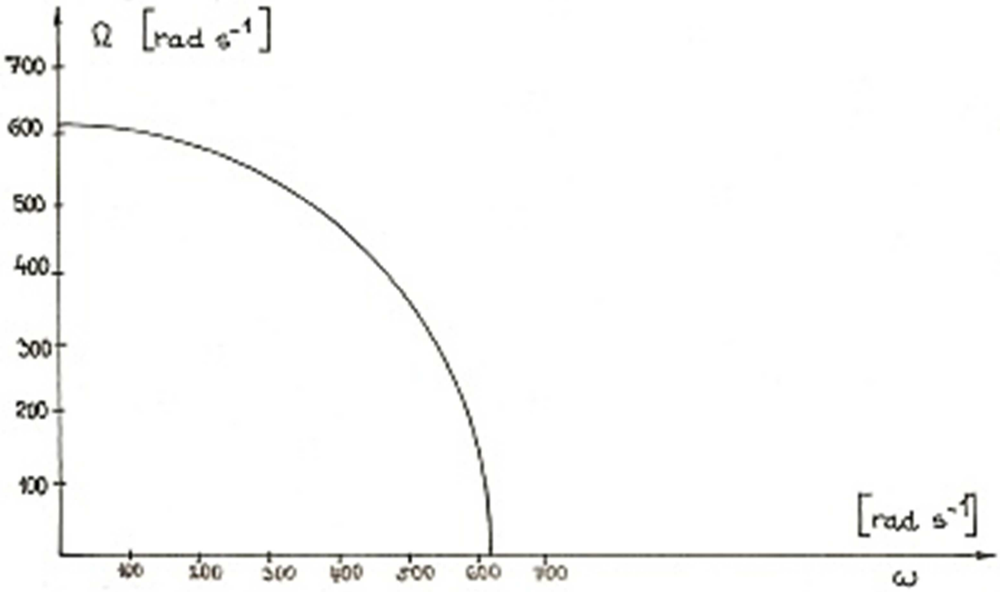

Figure 5. *The natural frequency dependence of discrete model´s relative transverse vibration (shown in Figure 4) on the angular velocity of rotation.*

4. Speed Resonance Analytical Solution of Propeller Shaft's Test Model

In the reference number [1], there was derived an equation of an one-dimensional continuum oscillating in a plane in the following form:

$$\frac{\partial^4 y}{\partial x^4} - \frac{\rho S r^2}{4EJ} \cdot \frac{\partial^4 y}{\partial x^2 \partial t^2} - \frac{\rho S r^2 \omega^2}{4EJ} \cdot \frac{\partial^2 y}{\partial x^2} + + \frac{\rho S}{EJ} \cdot \frac{\partial^2 y}{\partial t^2} - \frac{\rho S \omega^2}{EJ} y = 0, \tag{11}$$

where is:

y = y(x,t).... instantaneous displacement of centre-section in the rotating plane

ρ [kgm^{-3}].... material density

S [m^{-2}].... cross sectional area

J [m^4].... geometrical moment of inertia about an axis perpendicular to the neutral axis

E [Pa].... modulus of elasticity in tension or compression

r [m].... radius of the circular cross section of the shaft

ω [$rads^{-1}$].... the angular velocity of rotation of the plane

0(x,y) around the axis x

The above equation of motion will be applied when calculating the natural frequencies of propeller shaft´s test model. This is a shaft of constant circular cross section in rigid bearings which does the bending vibration during rotation (see Figure 6).

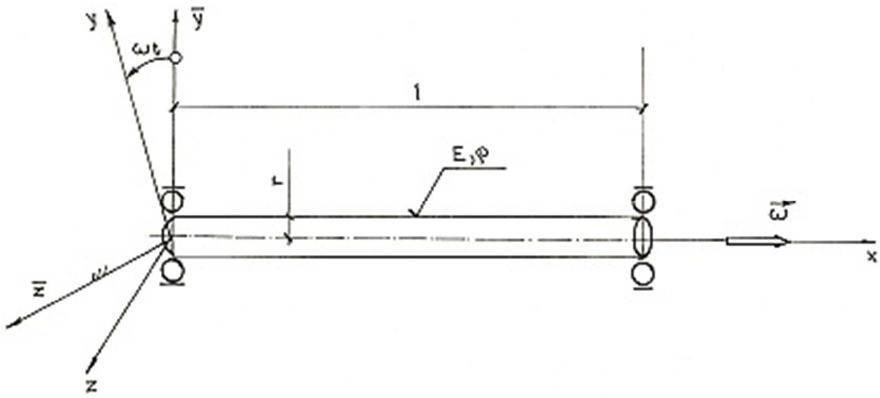

Figure 6. *Test model for calculation of propeller shaft´s speed resonance.*

Assume that each shaft section is held oscillating movement with an amplitude dependent on the place are of the time course the same for the whole shaft. Then we can look for solution in the following form:

$$y\,(x,t) = Y\,(x)\, e^{i\Omega t}, \tag{12}$$

where it is:

Y(x) … an amplitude of deflection on the x-coordinate

Ω [$\mathrm{rad\,s^{-1}}$] … the desired angular frequency oscillations of its own

The solution (12) is substituted for (11)

$$\left[Y^{IV}(x) + \frac{\rho S r^2 \Omega^2}{4EJ} Y^{II}(x) - \frac{\rho S r^2 \omega^2}{4EJ} Y^{II}\,(x) - - \frac{\rho S \Omega^2}{EJ}\, Y(x) - \frac{\rho S \omega^2}{EJ}\, Y(x)\right] e^{i\Omega t} = 0 \tag{11}$$

If the equation (13) has to be satisfied identically at each time, must apply:

$$Y^{IV}(x) + \,bY^{II}\,(x) - cY(x) = 0,$$

where

$$b = \frac{\rho S r^2}{4EJ}\,(\Omega^2 \,- \omega^2),\; c = \frac{\rho S}{EJ}\,(\Omega^2 + \,\omega^2). \tag{12}$$

Due to boundary conditions (the shaft is mounted on rigid bearings), we look for solution of the amplitude equation in the form:

$$Y(x) = Y \sin k_n\, x, \tag{13}$$

where

$$k_n = \frac{\pi n}{l},\; n = 1,2,..., \tag{14}$$

l is the length of the shaft. Then we substitute the envisaged solution (15) into the amplitude´s equation (14):

$$[k_n^4 - bk_n^2 - \,c]\; Y \sin k_n\, x = 0. \tag{15}$$

From the condition of nontrivial solution of the equation (17) we obtain a frequency equation of the form:

$$k_n^4 - bk_n^2 - c = 0. \tag{16}$$

From the equation (18) we obtain the relation for angular frequencies of shaft´s relative transverse oscillations in the form:

$$\Omega_n = \left(\frac{EJ}{\rho S}\right)^{\frac{1}{2}} \left\{\frac{\left(\frac{\pi n}{l}\right)^4 - \frac{\rho S \omega^2}{EJ}\left[1 - \left(\frac{\pi n r}{2l}\right)^2\right]}{1 + \left(\frac{\pi n r}{2l}\right)^2}\right\}^{\frac{1}{2}}. \tag{17}$$

For the first angular frequency, when focusing on long, slender shafts (l >> r), the term (19) is in simplified form:

$$\Omega = \left(\frac{EJ}{\rho S}\right)^{\frac{1}{2}} \left[\left(\frac{\pi}{l}\right)^4 - \frac{\rho S \omega^2}{EJ}\right]^{\frac{1}{2}}. \quad (20)$$

In case $\omega = 0$

$$\Omega = \frac{r}{2}\left(\frac{EJ}{\rho S}\right)^{\frac{1}{2}} \cdot \left(\frac{\pi}{l}\right)^2,$$

or, if we substitute for J, S

$$\Omega = \frac{r}{2}\left(\frac{E}{\rho}\right)^{\frac{1}{2}} \cdot \left(\frac{\pi}{l}\right)^2. \quad (21)$$

a known relationship to its angular frequency of beam´s transverse oscillations of constant circular cross-section on solid supports. Before we proceed to the quantification of the above relationships, analyze the impact of the angular velocity of rotation on its angular frequency of transverse oscillations of the shaft. We modify the expression (20) in the form:

$$\Omega^2 = a^2(b - c\omega^2), \quad (22)$$

where

$$a = \left(\frac{EJ}{\rho S}\right)^{\frac{1}{2}}, b = \left(\frac{\pi}{l}\right)^4, c = \frac{\rho S}{EJ} = \frac{1}{a^2}. \quad (23)$$

We modify the expression (22) to the form:

$$\frac{\Omega^2}{ba^2} + \frac{\omega^2}{\frac{b}{c}} = 1, \quad (24)$$

where

$$ba^2 = \left(\frac{\pi}{l}\right)^4 \cdot \frac{EJ}{\rho S} = R^2, \frac{b}{c} = \left(\frac{\pi}{l}\right)^4 \cdot \frac{EJ}{\rho S} = R^2.$$

Thus, the equation (24) can therefore be rewritten into the form:

$$\frac{\Omega^2}{R^2} + \frac{\omega^2}{R^2} = 1, \quad (28)$$

which is in the coordinate system of the $0(\omega, \mathbf{\Omega})$ a equation of a circle of radius:

$$R = \left(\frac{EJ}{\rho S}\right)^{\frac{1}{2}} \left(\frac{\pi}{l}\right)^2 = \frac{r}{2}\left(\frac{E}{\rho}\right)^{\frac{1}{2}} \left(\frac{\pi}{l}\right)^2. \quad (29)$$

The radius is equal to its angular frequency of shaft´s transverse oscillations at zero speed. The calculation is performed for

$E = 2{,}1.10^{11}$ [Pa], $\rho = 7{,}8.10^3$ [kg.m^{-3}], r = 0,0105 [m], l = 0,65 [m].

$$R = \frac{0.0105}{2}\left(\frac{2{,}1 \cdot 10^{11}}{7{,}8 \cdot 10^3}\right)^{\frac{1}{2}} \cdot \left(\frac{\pi}{0{,}65}\right)^2 = 636{,}35 \text{ [rad s}^{-1}\text{]}.$$

The functional dependence graph $\boldsymbol{\Omega} = \boldsymbol{\Omega}(\omega)$ for the considered test example is shown in Figure 7.

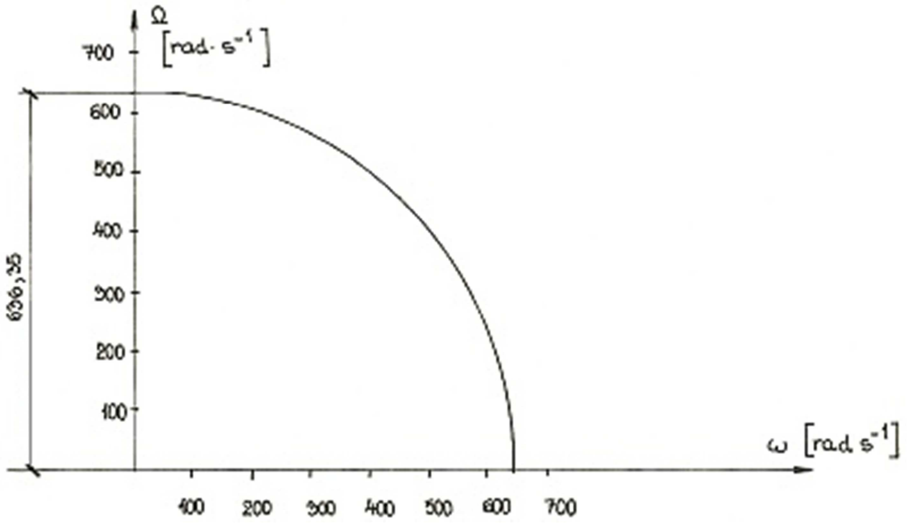

Figure 7. *The graph functional dependence of the angular frequency (**Ω**) lateral vibrations relative to the angular speed of rotation of the propeller shaft´s test model.*

5. Conclusions

As a result of the torque transmission through Hook´s joints, the connecting shaft is loaded by additional dynamic bending moments in the plane of the rotating propeller shaft. We can not accept the assumption that shaft deflection wheels, as in the case of a simple circular vibration of the continuum, but as a result of excitation rotating torque is obviously forced to hold the relative transversal oscillations. In the first step we formulate a mathematical model of transverse oscillations in the rotating plane. It is performed an analytical solution. On the test model, there is performed numerical solution that is used in the next step for tuning accurate models based on the transfer-matrix method and on the finite element method.

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