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An inventory model with Poisson demand and linear lost sales

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Abstract

In this article, we shall construct a lost sales inventory model with Poisson demand. Lost sales are linear increasing function of waiting time in the stockout period. Then, the optimal planning and shortage periods are determined such that the expected profit per unit time is maximized.There are two special cases are also discussed.One is the whole period is stockout, and the other is the whole period is non-stockout. And we also find the optimal supply cycle and optimal order quantity of goods such that expected profit per unit time is maximized.

Keywords

Inventory, Partially Backordered, Poisson Demand

1. Introduction

In the past works, many researchers considered the lost sales inventory model from different aspects and methods. For example, Montgomery, D. C. et al. [9] presents several single-echelon, single-item, static demand inventory models for situations in which, during the stockout period, a fraction b of the demand is backordered and the remaining fraction$1-b$ is lost forever. Schwartz, B. L.[11] considers an inventory model that the future demand is affected by stockouts. Caine, G. J. &Plaut, R. H. [4] generalized the Schwartz's model and determined the properties of the optimal policies. Ouyang, L. Y. et al. [10] then proposed an inventory model with partial lost-sales to effectivelyincrease investment and to reduce the lost-sales rate. Chiang, C. [7] proposed a dynamic programming model for periodic-review systems in which a replenishment cycle consists of a number of small periods(each of identical but arbitrary length) and holding and shortage costs are charged based on the ending inventory of small periods.

In the recent years, researchers construct the lost sales inventory model with stochastic demand. They developed the Markov model to represent the on-hand inventorylevel, e.g. Bijvank, M. &Johansen, S. G. [3] developed new models allowing constant lead time of any length when demand is compound Poisson. Axsäter, S. [1] considers a single-echelon inventory system with a warehouse facing compound Poissoncustomerdemand.Antonio A. R.&DeCroix, G. A. [8]explore the management of inventory for stochastic-demand systems, where the product's supply is randomly disrupted for periods of random duration, and demands that arrive when the inventory system is temporarily out of stock become a mix of backorders and lost sales. Chen, M.S. & Chen,Y.C. [5] consider an inventory problem basedon the assumption that lost sales depend on the waiting time when the whole period is stockout.Bijvank,M. et al. [2]also classify the lost-sales inventory models in the literature based on the characteristics of the inventory systems and review the proposedreplenishment policies.

The situation of shortage occurs frequently in the real world.When the customer arrives in the stockout period, he may leave or wait for a few days to get the item. The customer behavior to such stock-out occurrences appears to be rather complex. Many factors will affect his decision.

Wethink that the waiting time is one of the most important factors. The longer the waiting time is, the lesser the waiting willingness of the customer.Thus, to determine the suitable length of stockout period is important to the decision maker if shortage of goods is permitted.

Therefore, we shall construct the inventory model when the demand rate follows a Poisson distribution and the customer's waiting willingness is a decreasingfunction of waiting time. However,it is difficult to determine the inventory on hand when the demand is random, we intend to propose the conceptof expected order quantity and expected backordered quantity. Hence, we first determine the (planned) non-stockout period $[0,s]$ and the stockoutperiod $[s,t]$, then find expected orderquantity and the expected backordered quantity such that the expected profit per unit time is maximized.

2. Model Formulation

To formulate the mathematical model the following notation and assumptions are used throughout the article.

(1) k: The setup cost.
(2) v: The unit price.
(3) c: The unit cost of goods.
(4) h: The unit holding cost per unit time.
(5) $[0, t]$: The planning period.
(6) t: The length of planning period.
(7) $[0, s]$: The time interval we plan to be unshortage.
(8) $[s, t]$: The time interval we plan to be shortage.

A In $[0, s]$, there are two random phenomena should be considered:

(1) N_s: The total customers comes in the time interval $[0, s]$, and has a Poisson distribution with parameter λs.
(2) X: The time that the customer comes to purchase the goods in the time interval$[0, s]$. We say that it is the entering time of the customer. Here, we also assume that X_i is the purchasing time of the i^{th} customers, where i=1, 2, ⋯, N_s. We also assume that X_1, X_2, ⋯, X_s are independent and identically distributed with p.d.f. $f_X(x)=\frac{1}{s}$.

Therefore,

(1) Expected total holding cost H:

Since X_i is the purchasing time of the i^{th} customers, it means that the length of holding time for the i^{th} unit of goods is X_i. Hence, the total length of holding time for N_s units of goods is then

$$L = X_1 + X_2 + \cdots + X_{N_s}.$$

The distribution of L is the so-called compound Poisson distributionwith parameter λs.

Since the holding cost per unit time is h, therefore the total holding cost is h·L, and the expected total holding cost is given by

$$H=E[h\cdot L]=hE[E[L|N_s]]=h\lambda s\cdot E[X]=\frac{h\lambda s^2}{2}. \quad (1)$$

(2) Expected total purchasing cost P:

Since the total quantity of demand is N_s, so the total purchasing cost is c· N_s , and the expected total purchasing cost is

$$P=\sum_{n=0}^{\infty} c\cdot nP(N_s=n)=cE[N_s]=c\lambda s. \quad (2)$$

(3) Expected total revenue R:

Since the total quantity of demand is N_s, so the total revenue is v· N_s, and the expected total revenue is

$$R=\sum_{n=0}^{\infty} v\cdot nP(N_s=n)=vE[N_s]=v\lambda s.$$

B In $[s, t]$, there are three random phenomena should be considered:

(1) N_{t-s}: The total demand in the time interval $[s, t]$.And N_{t-s} has a Poisson distribution with parameter $\lambda(t-s)$.
(2) Y: The entering time of the customer who comes to purchase the goods in the time interval $[s, t]$. Since the period is shortage, so when the customer enters at time y, $y\in [s,t]$, then he must wait until t to get the goods. The waiting willingness of the customer depends on the length of waiting time. The larger the waiting time is, the lesser the waiting willingness or possibility.
(3) ω: The maximum length of time that the customer is willing to wait.

If the customer arrives at time Y=y, then the length of waiting time is $t-y$. Here we assume that the probability of the waiting willingness of the customer is a linear decreasing function of waiting time and is given by

$$p=1-\frac{t-y}{\omega},\ \omega\geq t-s. \quad (4)$$

Let θ be the expected probability for any customer arrive at time Y, then we have

$$\theta=E\left[1-\frac{t-Y}{\omega}\right]=1-\frac{t-s}{2\omega} \quad (5)$$

Since the total number of customers arrive in the time interval $[s, t]$ is N_{t-s}, so the proportion of customers will wait until t is θN_{t-s} and the expected profit is given by

$$E[(v-c)\theta N_S]=(v-c)\lambda(t-s)\left[1-\frac{t-s}{2\omega}\right].(6)$$

Together with (1), (2), (3), (6) and setup cost, the total profit is then given by

$$(v-c)\lambda s-\frac{h\lambda s^2}{2}+(v-c)\lambda(t-s)\left[1-\frac{t-s}{2\omega}\right]-k$$

Therefore, the expected profit per unit time is then

$$\frac{(v-c)\lambda s}{t}-\frac{h\lambda s^2}{2t}+\frac{(v-c)\lambda}{t}\left[t-s-\frac{(t-s)^2}{2\omega}\right]-\frac{k}{t} \quad (7)$$

3. Optimal Solution

Our objective is to find s and t such that the expected profit per unit time is maximized, i.e.,

$$\mathrm{Max}_{0\leq s\leq t}\, \mathrm{A}(s,t) \tag{8}$$

where

$$\mathrm{A}(s,t)=\frac{(v-c)\lambda s}{t}-\frac{h\lambda s^2}{2t}+\frac{(v-c)\lambda}{t}\left[t-s-\frac{(t-s)^2}{2\omega}\right]-\frac{k}{t}$$

To find the maximum of function A, differentiate A(s,t) with respect to s and set it equals to zero, we have

$$\frac{\partial \mathrm{A}(s,t)}{\partial s}=\frac{(v-c)\lambda}{t}-\frac{h\lambda s}{t}-\frac{(v-c)\,\lambda}{t}+\frac{(v-c)\,\lambda(t-s)}{t\omega}=0. \tag{9}$$

Then,

$$s=\frac{v-c}{h\omega+v-c}t. \tag{10}$$

And, the second order differentiation of A with respect to s is

$$\frac{\partial^2\mathrm{A}}{\partial s^2}=-\frac{h\lambda}{t}-\frac{(v-c)\,\lambda}{t\omega}<0$$

This means that s=s(t) in (10) will maximize A(s, t) for any given t. Thus, (8) canbe rewritten as follows:

$$\max_{0\leq s\leq t}\mathrm{A}(s,t)=\max_{0\leq s(t)\leq t}\mathrm{A}(s(t),t) \tag{11}$$

Then, differentiate A(s(t), t) with respect to t, we have

$$\frac{d\,\mathrm{A}(s(t),t)}{dt}=-\frac{1}{2}\frac{2h\lambda s(t)s'(t)t-h\lambda s^2(t)}{t^2}$$
$$-\frac{(v-c)\,\lambda}{2\omega}\frac{2(t-s(t))(1-s'(t))t-(t-s(t))^2}{t^2}+\frac{k}{t^2} \tag{12}$$

Since $s'(t)=\frac{v-c}{h\omega+v-c}$ and by (10), then (12) becomes

$$\frac{d\,\mathrm{A}(s(t),t)}{dt}=-\frac{h\lambda}{2}\left(\frac{v-c}{h\omega+v-c}\right)^2-\frac{(v-c)\,\lambda}{2\omega}\left(\frac{h\omega}{h\omega+v-c}\right)^2+\frac{k}{t^2}.$$

Set $\frac{d\,\mathrm{A}(s(t),t)}{dt}$=0, then we have

$$t=\sqrt{\frac{2k(h\omega+v-c)}{h\lambda(v-c)}}. \tag{13}$$

And the second order differentiation of A(s(t), t) with respect to t is given by

$$\frac{d^2\mathrm{A}(s(t),t)}{dt^2}=-\frac{2k}{t^3}<0.$$

This implies that $t^*=\sqrt{\frac{2k(h\omega+v-c)}{h\lambda(v-c)}}$ given in (13) is the optimal solution of (11). And,from (10), we have

$$s^*=\frac{v-c}{h\omega+v-c}t^*=\frac{v-c}{h\omega+v-c}\sqrt{\frac{2k(h\omega+v-c)}{h\lambda(v-c)}}=\sqrt{\frac{2k(v-c)}{h\lambda(h\omega+v-c)}}. \tag{14}$$

Therefore, (s^*, t^*) given in (13) and (14) is the optimal solution of (8).

4. Conclusion

From (13) and (14), we know that the total number of customers arrive in the time interval [0, s^*] and [s^*,t^*] are N_{s^*} and $N_{t^*-s^*}$,respectively. And since N_{s^*} and $N_{t^*-s^*}$ are Poisson distribution with parameter λs^* and $\lambda(t^*-s^*)$, respectively. Thus, the expected order quantity ofgoods is

$$q_{s^*}=E[N_{s^*}]=\lambda s^*=\sqrt{\frac{2k\lambda(v-c)}{h(h\omega+v-c)}} \tag{15}$$

and expected backordered quantity of goods is

$$q_{t^*-s^*}=E[N_{t^*-s^*}]=\lambda(t^*-s^*)=\sqrt{\frac{2k\lambda(h\omega+v-c)}{h(v-c)}}-\sqrt{\frac{2k\lambda(v-c)}{h(h\omega+v-c)}} \tag{16}$$

The q_{s^*} and$q_{t^*-s^*}$ derived in (15) and (16) will be the optimal solution of (8). We can interpret the meaning of q_{s^*} and $q_{t^*-s^*}$ as follows:

If we order q_{s^*} at time 0 and reorder at t^*, then q_{s^*} may use up less than s^* because of the demand is random. And, in this case, the backordered quantity of goods willincrease. Conversely, if q_{s^*} use up greater than s^* then the backordered quantity of goods will decrease. But, in the long run, the expected backordered quantity of goods will be equal to $q_{t^*-s^*}$ as in (16).

In fact, if the demand rate per unit time is, λ, a constant and other assumptions remain the same, then the model becomes a deterministic one.

Hence, the optimal order quantity of goods and backordered quantity of goods are also q_{s^*} and $q_{t^*-s^*}$, respectively.

There is an unknown parameter ω(the maximum length of time that the customer is willing to wait) should be estimated. We can estimate it by the past experience. Suppose that we have the following dataof the n customers:

$$X_1,\ X_2,\ \cdots,\ X_n$$

where X_i is the maximum length of time that the i^{th} customer is willing to wait. Then we can estimate ω as follows:

$$\hat{\omega}=\max\{X_i\},$$

which is the maximum likelihood estimator of ω.

Furthermore, there are two special cases should be considered:

Case 1.s=0: this means that the whole period is stockout. Then, the expected profit functionis reduced to

$$\mathrm{A}(t)=(v-c)\lambda-(v-c)\lambda\frac{t}{2\omega}-\frac{k}{t}$$

It is easy to see that

$$\mathrm{A}'(t)=-\frac{(v-c)\lambda}{2\omega}+\frac{k}{t^2}=0$$

then

$$t^*=\sqrt{\frac{2k\omega}{(v-c)\,\lambda}} \tag{17}$$

and$A''(t)$=$-\frac{2k}{t^3}$<0. Thus, A(t) has a maximum at t^*.

In real life, sometimes goods or services cannot be provided immediately or need batch processing. For example, how frequent is the training program offered? Thesekind of problems are out-of-stock in the whole period and the goods or services supply at the end of period.

Hence, t^*, derive in (17), is then the optimal goods or services supply cycle.

Case 2.s=t: the expected profit function is then reducedto

$$A(s)=(v-c)\lambda-\frac{h\lambda s}{2}-\frac{k}{s}$$

It is easy to see that

$$A'(s)=-\frac{h\lambda}{2}+\frac{k}{s^2}=0,$$

and

$$s^*=\sqrt{\frac{2k}{h\lambda}}$$

and$A''(s)$=$-\frac{2k}{s^3}$<0. Hence, A(s) has a maximum at s^*. Thus,

$$q_{s^*}=E[N_{s^*}]=\lambda s^*=\sqrt{\frac{2k\lambda}{h}}. \tag{18}$$

q_{s^*}, given in (18), is the same as the economic order quantity of the traditional EOQ model, however, we say that q_{s^*} is the expected economic order quantity(EEOQ) in this model[6]. This means that, in the long run, the average order quantity of goods is q_{s^*} will maximize the profit.

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