# Certain class of meromorphic univalent functions defined by an Erdelyi-Kober type integral operator 

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#### Abstract

In this paper, we investigate interesting subordination properties for certain subclasses of meromorphic analytic and univalent functions in the puncture unit disc which are defined here by means of new linear operator. Further, few interesting special cases and examples are obtained for an appropriate choices of the parameters and the corresponding functions.


## Keywords

Meromorphic Function, Convex Function, Convolution, Differential Subordination

## 1. Introduction

Let denote by $H(\mathrm{U})$ the space of all analytical functions in the unit disc $\mathrm{U}=\{z \in C:|z|<1\}$, and for $a \in C, n \in \mathrm{~N}^{*}$, we denote

$$
H[a, n]=\left\{f \in H(\mathrm{U}): f(z)=a+a_{n} z^{n}+\cdots\right\} .
$$

Let denote the class of functions

$$
A_{n}=\left\{f \in H(\mathrm{U}): f(z)=z+a_{n+1} z^{n+1}+\cdots\right\},
$$

and let $A \equiv A_{1}$.
If $f, F \in H(\mathrm{U})$ and $F$ is univalent in U we say that the function $f$ is subordinate to $F$, written $f(z) \prec F(z)$, if $f(0)=F(0)$ and $f(\mathrm{U}) \subseteq F(\mathrm{U})$
Let $\phi(r, s ; z): \mathrm{C}^{3} \times U \rightarrow \mathrm{C}$ and $h(z)$ be univalent in U . If $p(z)$ is analytic in $U$ and satisfies the second order differential subordination:

$$
\begin{equation*}
\phi\left(p(z), z p^{\prime}(z), z^{2} p^{\prime \prime}(z) ; z\right) \prec h(z) \tag{1.1}
\end{equation*}
$$

then $p(z)$ is a solution of the differential subordination (1.1). The univalent function $q(z)$ is called a dominant of the
solutions if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.1). A univalent dominant $\hat{q}$ that satisfies $\hat{q} \prec q$ for all dominants of (1.1) is called the best dominant (see [13]).

Miller et al. [15] investigated some subordination theorems involving certain integral operators for analytic functions in U (see also [3, 16]).

Let $\Sigma$ be the class of functions of the form:

$$
\begin{equation*}
\left.f(z)=\frac{1}{z}+\sum_{n=1}^{\infty} a_{n} z^{n} \quad(n \in \mathrm{~N}=\{1,2, \ldots\}\}\right), \tag{1.2}
\end{equation*}
$$

which are analytic and univalent in the puncture open unit disc $U^{*}=\{z: z \in \mathrm{C}$ and $0<|z|<1\}=U \backslash\{0\}$.

Let $m \in \mathrm{Z}=\{0, \pm 1, \pm 2, \ldots\}$ and for $\ell>0, \lambda>0$ the operator $L^{m}(\lambda, \ell) f(z): \Sigma \rightarrow \Sigma$ (see [6] and [7] with $p=1$ ) be defined as follows:

$$
\begin{gathered}
L^{m}(\lambda, \ell) f(z)=f(z), \quad m=0 \\
L^{m}(\lambda, \ell) f(z)=\frac{\ell}{\lambda} z^{-1-\frac{1}{\lambda}} \int_{0}^{z} t^{\frac{1}{\lambda}} L^{m+1}(\lambda, \ell) f(t) d t \\
(m=-1,-2, \ldots ; z \in c),
\end{gathered}
$$

$$
\begin{gather*}
L^{m}(\lambda, \ell) f(z)=\frac{\lambda}{\ell} z^{-\frac{\ell}{\lambda}} \frac{d}{d z}\left(z^{\frac{1}{\lambda}+1} L^{m-1}(\lambda, \ell) f(z)\right)  \tag{1.3}\\
\left(m=1,2, \ldots ; z \in U^{*}\right) .
\end{gather*}
$$

Let $A>0, a, c \in \mathrm{C}$, be such that $\operatorname{Re}(a)>A$, $\operatorname{Re}(c-a) \geq 0$, modified an Erdelyi-Kober type integral operator (see, [8], Ch. 5), we define $\tilde{I}_{A}^{a, c}: \Sigma \rightarrow \Sigma$ by

$$
\begin{equation*}
\tilde{I}_{A}^{a, c} f(z)=\frac{\Gamma(c-A)}{\Gamma(a-A) \Gamma(c-a)} \int_{0}^{1} t^{a-1}(1-t)^{c-a-1} f\left(z t^{A}\right) d t \tag{1.4}
\end{equation*}
$$

and

$$
\tilde{I}_{A}^{a, a} f(z)=f(z) .
$$

Now the operator $L_{\lambda, \ell}^{m}(a, c, A): \Sigma \rightarrow \Sigma$ is defined by

$$
L_{\lambda, \ell}^{m}(a, c, A) f(z)=L^{m}(\lambda, \ell) \widehat{I}_{A}^{a, c} f(z)=\widetilde{I}_{A}^{a, c} L^{m}(\lambda, \ell) f(z),
$$

whose series expansion take the form

$$
\begin{align*}
& L_{\lambda, \ell}^{m}(a, c, A) f(z) \\
& \begin{array}{c}
=\frac{1}{z}+\frac{\Gamma(c-A)}{\Gamma(a-A)} \sum_{n=1}^{\infty}\left[\frac{\ell+\lambda(n+1)}{\ell}\right]^{m} \frac{\Gamma(a+n A)}{\Gamma(c+n A)} a_{n} z^{n} \\
(m \in \mathrm{Z} ; \ell>0 ; \lambda>0 ; A>0 ; a, c \in \mathrm{C} ; \\
\operatorname{Re}(c-a)>0 ; \operatorname{Re}(a)>A) .
\end{array}
\end{align*}
$$

We note that this new class of operator include in turn several operators, we may point out here some special cases of them which can be found in
(i) $L_{1,1}^{\alpha}(a, a, A) f(z)=P^{\alpha} f(z)$ (see Aqlan et al. [2], with $p=1$ );
(ii) $L_{l, \beta}^{\alpha}(a, a, A) f(z)=P_{\beta}^{\alpha} f(z)$ (see Lashin [9]).
(iii) $\quad L_{1, \ell}^{m}(a, a, A) f(z)=I(m, \ell) f(z)(m>0)$ (see Cho et al. $[4,5])$;
(iv) $L_{\lambda, 1}^{m}(a, a, A) f(z)=D_{\lambda}^{m} f(z)(m>0) \quad$ (see Al-Oboudi and Al-Zkeri [1], with $p=1$ );
(v) $L_{1,1}^{m}(a, a, A) f(z)=I^{m} f(z)(m>0)$ (see Uralegaddi and Somanatha [18]).

Also, we note that

$$
\begin{align*}
& L_{\lambda, \ell}^{m}(a, a, A) f(z)=L^{m}(\lambda, \ell) f(z) \\
& \text { and } L_{\lambda, \ell}^{0}(a, c, A) f(z)=\tilde{I}_{A}^{a, c} f(z) . \tag{1.6}
\end{align*}
$$

From (1.5), it is easy to verify that

$$
\begin{align*}
& z\left(L_{\lambda, \ell}^{m}(a, c, A) f(z)\right) \\
= & \frac{\ell}{\lambda} L_{\lambda, \ell}^{m+1}(a, c, A) f(z)-\left(1+\frac{\ell}{\lambda}\right) L_{\lambda, \ell}^{m}(a, c, A) f(z) . \tag{1.7}
\end{align*}
$$

and

$$
\begin{align*}
& z\left(L_{\lambda, \ell}^{m}(a, c, A) f(z)\right) \\
= & \left(\frac{a-A}{A}\right) L_{\lambda, \ell}^{m}(a+1, c, A) f(z)-\frac{a}{A} L_{\lambda, \ell}^{m}(a, c, A) f(z) \tag{1.8}
\end{align*}
$$

In this paper, we drive interesting subordination results for certain subclasses of meromorphic analytic and univalent functions in the puncture unit disc which are defined here by means of new linear operator $L_{\lambda, \ell}^{m}(a, c, A)$.

## 2. Preliminaries

To prove our main results, we will need the following definitions and lemmas presented in this section.

A function $L(z ; t): \mathrm{U} \times[0,+\infty) \rightarrow C$ is called a subordination (or a Loewner) chain if $L(; ; t)$ is analytic and univalent in U for all $t \geq 0$, and $L(z ; s) \prec L(z ; t)$ when $0 \leq s \leq t$.

The next well-known lemma gives a sufficient condition so that the $L(z ; t)$ function will be a subordination chain.

Lemma 1. [17, p. 159] Let $L(z ; t)=a_{1}(t) z+a_{2}(t) z^{2}+\ldots$, with $a_{1}(t) \neq 0$ for all $t \geq 0$ and $\lim _{t \rightarrow+\infty}\left|a_{1}(t)\right|=+\infty$. Suppose that $L(\cdot ; t)$ is analytic in U for all $t \geq 0, L(z ; \cdot)$ is continuously differentiable on $[0,+\infty)$ for all $z \in \mathbb{U}$. If $L(z ; t)$ satisfies

$$
\operatorname{Re}\left[z \frac{\partial L / \partial z}{\partial L / \partial t}\right]>0, z \in \mathrm{U}, t \geq 0
$$

and

$$
|L(z ; t)| \leq K_{0}\left|a_{1}(t)\right|,|z|<r_{0}<1, t \geq 0
$$

for some positive constants $K_{0}$ and $r_{0}$, then $L(z ; t)$ is a subordination chain.

We denote by $K(\alpha), \alpha<1$, the class of convex functions of order $\alpha$ in the unit disk U , i.e.

$$
K(\alpha)=\left\{f \in A: \operatorname{Re}\left[1+\frac{z f^{\prime \prime}(z)}{f^{\prime}(z)}\right]>\alpha, z \in \mathrm{U}\right\}
$$

In particular, the class $K \equiv K(0)$ represents the class of convex (and univalent) functions in the unit disc.

Lemma 2.[11], [13, Theorem 2.3i, p. 35] Suppose that the function $H: C^{2} \rightarrow C$ satisfies the condition

$$
\operatorname{Re} H(i s, t) \leq 0
$$

for all $s, t \in C$ with $t \leq-n\left(1+s^{2}\right) / 2$, where $n$ is a positive integer. If the function $p(z)=1+p_{n} z^{n}+\ldots$ is analytic in U and

$$
\operatorname{Re} H\left(p(z), z p^{\prime}(z)\right)>0, z \in \mathrm{U}
$$

then $\operatorname{Re} p(z)>0, z \in \mathrm{U}$.
The next result deals with the solutions of the Briot--Bouquet differential equation, and more general forms of the following lemma may be found in [12, Theorem 1].

Lemma 3.[12] Let $\beta, \gamma \in C$ with $\beta \neq 0$ and let $h \in H(\mathrm{U})$, with $h(0)=c$. If $\operatorname{Re}[\beta h(z)+\gamma]>0, z \in \mathrm{U}$, then the solution of the differential equation

$$
q(z)+\frac{z q^{\prime}(z)}{\beta q(z)+\gamma}=h(z),
$$

with $q(0)=c$, is analytic in $U$ and satisfies $\operatorname{Re}[\beta q(z)+\gamma]>0, z \in \mathrm{U}$.

As in [14], let denote by $\mathbf{Q}$ the set of functions $f$ that are analytic and injective on $\overline{\mathbf{U}} \backslash E(f)$, where

$$
E(f)=\left\{\zeta \in \partial \mathrm{U}: \lim _{z \rightarrow \zeta} f(z)=\infty\right\}
$$

and such that $f^{\prime}(\zeta) \neq 0$ for $\zeta \in \partial \mathrm{U} \backslash E(f)$.
Lemma 4.[14, Theorem 7] Let $q \in H[a, 1]$, let $\chi: C^{2} \rightarrow C \quad$ and $\quad$ set $\quad \chi\left(q(z), z q^{\prime}(z)\right) \equiv h(z) \quad$. If $L(z ; t)=\chi\left(q(z), t z q^{\prime}(z)\right)$ is a subordination chain and $p \in H[a, 1] \cap Q$, then

$$
h(z) \prec \chi\left(p(z), z p^{\prime}(z)\right) \text { implies } q(z) \prec p(z) .
$$

Furthermore, if $\chi\left(q(z), z q^{\prime}(z)\right)=h(z)$ has a univalent solution $q \in \mathbb{Q}$, then $q$ is the best subordinant.

Like in [11] and [13], let $\Omega \subset C, q \in \mathrm{Q}$ and $n$ be a positive integer. Then, the class of admissible functions $\Psi_{n}[\Omega, q]$ is the class of those functions $\psi: C^{3} \times \mathrm{U} \rightarrow C$ that satisfy the admissibility condition

$$
\psi(r, s, t ; z) \notin \Omega,
$$

whenever $r=q(\zeta), s=m \zeta q^{\prime}(\zeta), \operatorname{Re} \frac{t}{s}+1 \geq m \operatorname{Re}\left[\frac{\zeta q^{\prime}(\zeta)}{q^{(\zeta)}}+1\right]$, $z \in \mathbf{U}, \zeta \in \partial \mathbf{U} \backslash E(q)$ and $m \geq n$. This class will be denoted by $\Psi_{n}[\Omega, q]$.

We write $\Psi[\Omega, q] \equiv \Psi_{1}[\Omega, q]$. For the special case when $\Omega \neq C$ is a simply connected domain and $h$ is a conformal mapping of U onto $\Omega$, we use the notation $\Psi_{n}[h, q] \equiv \Psi_{n}[\Omega, q]$.

Remark 1. If $\psi: C^{2} \times \mathbf{U} \rightarrow C$, then the above defined admissibility condition reduces to

$$
\psi\left(q(\zeta), m \zeta q^{\prime}(\zeta) ; z\right) \notin \Omega
$$

when $z \in \mathrm{U}, \quad \zeta \in \partial \mathrm{U} \backslash E(q)$ and $m \geq n$.
Lemma 5.[11], [13] Let $h$ be univalent in $U$ and
$\psi: C^{3} \times \mathrm{U} \rightarrow C$. Suppose that the differential equation

$$
\psi\left(q(z), z q^{\prime}, z^{2} q^{\prime \prime}(z) ; z\right)=h(z)
$$

has a solution $q$, with $q(0)=a$, and one of the following conditions is satisfied:
(i) $q \in \mathrm{Q}$ and $\psi \in \Psi[h, q]$
(ii) $q$ is univalent in U and $\psi \in \Psi\left[h, q_{\rho}\right]$, for some $\rho \in(0,1)$, where

$$
q_{\rho}(z)=q(\rho z), \text { or }
$$

(iii) $q$ is univalent in U and there exists $\rho_{0} \in(0,1)$ such that $\psi \in \Psi\left[h_{\rho}, q_{\rho}\right]$

$$
\text { for all } \rho \in\left(\rho_{0}, 1\right) \text {, where } h_{\rho}(z)=h(\rho z) \text { and } q_{\rho}(z)=q(\rho z) \text {. }
$$

$$
\text { If } \quad p(z)=a+a_{1} z+\ldots \in H(\mathrm{U})
$$

and
$\psi\left(p(z), z p^{\prime}, z^{2} p^{\prime \prime}(z) ; z\right) \in H(\mathrm{U})$, then

$$
\psi\left(p(z), z p^{\prime}, z^{2} p^{\prime \prime}(z) ; z\right) \prec h(z) \text { implies } p(z) \prec q(z)
$$

and $q$ is the best dominant.

## 3. Main Results

Unless otherwise mentioned, we assume throughout this paper that $0 \leq \beta \leq 1, m \in \mathrm{Z}, \ell>0, \lambda>0, \frac{\ell}{\lambda}>1, A>0, a, c \in \mathrm{C}$, $\operatorname{Re}(c-a)>0$ and $\operatorname{Re}(a)>A$.

We begin by proving the following subordination theorem: Theorem 1. Let

$$
\begin{equation*}
\delta=\frac{\ell(a-A)}{(1-\beta) \lambda(a-A)+\beta A \ell} . \tag{3.1}
\end{equation*}
$$

be such that $\operatorname{Re}(\delta) \geq 1, \quad$ and for a given function $g \in \Sigma$,

$$
\begin{align*}
& \varphi(z) \\
& =z^{2}\left\{(1-\beta) L_{\lambda, \ell}^{m+1}(a, c, A) g(z)+\beta L_{\lambda, \ell}^{m}(a+1, c, A) g(z)\right\}, \tag{3.2}
\end{align*}
$$

satisfy

$$
\begin{equation*}
\operatorname{Re}\left[1+\frac{z \varphi^{\prime \prime}(z)}{\varphi^{\prime}(z)}\right]>-\rho(z \in \mathrm{U}) \tag{3.3}
\end{equation*}
$$

where, $\rho=0$ if $\operatorname{Re}(\boldsymbol{\delta})=1$ and for $\operatorname{Re}(\boldsymbol{\delta})>1$,

$$
\rho \leq\left\{\begin{array}{lr}
\frac{\operatorname{Re}(\delta)-1}{2} & 1<\operatorname{Re}(\boldsymbol{\delta})<2  \tag{3.4}\\
\frac{1}{2(\operatorname{Re}(\delta)-1)} & \operatorname{Re}(\boldsymbol{\delta})>2
\end{array}\right.
$$

and

$$
\begin{equation*}
(\operatorname{Im}(\delta))^{2} \leq(\operatorname{Re}(\delta)-1-2 \rho)\left(\frac{1}{2 \rho}-\operatorname{Re}(\delta)+1\right) \tag{3.5}
\end{equation*}
$$

The equality in (3.4) and (3.5) occur only when $\operatorname{Im}(\boldsymbol{\delta})=0$. If $f \in \Sigma$ such that

$$
z^{2}\left\{(1-\beta) L_{\lambda, \ell}^{m+1}(a, c, A) f(z)+\beta L_{\lambda, \ell}^{m}(a+1, c, A) f(z)\right\} \prec \varphi(z), \text { (3.6) }
$$

then

$$
\begin{equation*}
z^{2} L_{\lambda, \ell}^{m}(a, c, A) f(z) \prec z^{2} L_{\lambda, \ell}^{m}(a, c, A) g(z) \tag{3.7}
\end{equation*}
$$

and the function $z^{2} L_{\lambda, \ell}^{m}(a, c, A) g(z)$ is the best dominant.
Proof. Let

$$
F(z)=z^{2} L_{\lambda, \ell}^{m}(a, c, A) f(z), \quad G(z)=z^{2} L_{\lambda, \ell}^{m}(a, c, A) g(z), \text { (3.8) }
$$

By hypothesis we first show that the function $G(z)$ is convex univalent. Let

$$
\begin{equation*}
q(z)=1+\frac{z G^{\prime \prime}(z)}{G^{\prime}(z)}(z \in U) \tag{3.9}
\end{equation*}
$$

Using (1.7) and (1.8) for $g(z) \in \Sigma$, we get

$$
\begin{equation*}
\varphi(z)=\left(1-\frac{1}{\delta}\right) G(z)+\frac{z G^{\prime}(z)}{\delta} \tag{3.10}
\end{equation*}
$$

where $\delta$ is given by (3.1). Differentiating (3.10), and using (3.9) we have

$$
\frac{\varphi^{\prime}(z)}{G^{\prime}(z)}=\left(1-\frac{1}{\delta}\right)+\frac{q(z)}{\delta}
$$

which in differentiating again, and using (3.9) we have

$$
\begin{equation*}
1+\frac{z \varphi^{\prime \prime}(z)}{\varphi^{\prime}(z)}=q(z)+\frac{z q^{\prime}(z)}{q(z)+\delta-1}=h(z) \tag{3.11}
\end{equation*}
$$

From (3.3) and (3.4) we have

$$
\operatorname{Re}[h(z)+\delta-1]>0
$$

and by using Lemma 3 we conclude that the differential equation (3.11) has a solution $q \in H(\mathrm{U})$, with $q(0)=h(0)=1$.

Now we will use Lemma 2 to prove that, under our assumption, the inequality

$$
\begin{equation*}
\operatorname{Re} q(z)>0,(z \in \mathrm{U}) \tag{3.12}
\end{equation*}
$$

holds. Let us put

$$
\begin{equation*}
H(u, v)=u+\frac{v}{u+\delta-1}+\rho \tag{3.13}
\end{equation*}
$$

where $\rho$ is given by (3.3). From the assumption (3.1), according to (3.11), we obtain

$$
\begin{equation*}
\operatorname{Re} H\left(q(z), z q^{\prime}(z)\right)>0 \quad(z \in \mathrm{U}) \tag{3.14}
\end{equation*}
$$

and we proceed to show that $\operatorname{Re} H(i s, t) \leq 0$ for all $s, t \in C$, with $t \leq-\left(1+s^{2}\right) / 2$. From (3.13), we have

$$
\begin{aligned}
\operatorname{Re} H(i s, t) & =\operatorname{Re}\left[i s+\frac{t}{i s+\delta-1}+\rho\right] \\
& =\frac{(\operatorname{Re}(\delta)-1) t}{|i s+\delta-1|^{2}}+\rho \leq-\frac{E(s)}{2|i s+\delta-1|^{2}}
\end{aligned}
$$

where

$$
\begin{equation*}
E(s)=(\operatorname{Re}(\delta)-1)(1+s)^{2}-2 \rho|i s+\delta-1|^{2} \tag{3.15}
\end{equation*}
$$

which on taking $\operatorname{Re}(\delta)=1$ and $\rho=0$, give $E(s)=0$, and $(\operatorname{Re}(\delta)-1)=u, \quad \operatorname{Im}(\delta)=v$, we have

$$
E(s)=(u-2 \rho) s^{2}-4 v \rho s+u-2 \rho\left(u^{2}+v^{2}\right)
$$

If $v=0$, from (3.3), we get

$$
E(s)=(u-2 \rho) s^{2}+u(1-2 \rho u) \geq 0 .
$$

If $v \neq 0$, from (3.4), we get

$$
\begin{align*}
E(s)= & (u-2 \rho)\left(s-\frac{2 v \rho}{(u-2 \rho)}\right)^{2} \\
& +u\left[1-2 \rho\left(u+\frac{v^{2}}{(u-2 \rho)}\right)\right] \geq 0 \tag{3.16}
\end{align*}
$$

From condition (3.5). Thus, $E(s) \geq 0$ for all $s \in \mathrm{R}$. Hence, from (3.15) and (3.16), we have $\operatorname{Re} H(i s, t) \leq 0$ for all $s, t \in R$, with $t \leq-\left(1+s^{2}\right) / 2$. Form (3.14), according to Lemma 2, we deduce that the inequality (3.12) holds, hence $G \in K$, that is $G$ a convex (and univalent) function in the unit disc, hence the following well-known growth and distortion sharp inequalities (see [10]) are true:

$$
\begin{aligned}
& \frac{r}{1+r} \leq|G(z)| \leq \frac{r}{1-r}, \text { if }|z| \leq r, \\
& \frac{1}{(1+r)^{2}} \leq\left|G^{\prime}(z)\right| \leq \frac{1}{(1-r)^{2}}, \text { if }|z| \leq r .
\end{aligned}
$$

If we let

$$
\begin{equation*}
L(z ; t)=\left(1-\frac{1}{\delta}\right) G(z)+\frac{(1+t)}{\delta} z G^{\prime}(z) \quad(z \in U ; t \geq 0) \tag{3.17}
\end{equation*}
$$

from (3.17) we have $L(z ; 0)=\varphi(z)$. Denoting $L(z ; t)=a_{1}(t) z+\ldots$, then

$$
\begin{equation*}
a_{1}(t)=\frac{\partial L(0 ; t)}{\partial z}=\left(1+\frac{t}{\delta}\right) G^{\prime}(0)=\left(1+\frac{t}{\delta}\right) \neq 0 \quad(t \geq 0) \tag{3.18}
\end{equation*}
$$

hence $\lim _{t \rightarrow+\infty}\left|a_{1}(t)\right|=+\infty$, we obtain $a_{1}(t) \neq 0, \quad \forall t \geq 0$. From (3.17) we may easily deduce the equality

$$
\operatorname{Re}\left[z \frac{\partial L / \partial z}{\partial L / \partial t}\right]=\operatorname{Re}(\delta)-1+(1+t) \operatorname{Re}\left(1+\frac{z G^{\prime \prime}(z)}{G^{\prime}(z)}\right)
$$

Using the inequality (3.12) together with the assumptions
$\operatorname{Re}(\delta)>1$, the above relation yields that

$$
\operatorname{Re}\left[z \frac{\partial L / \partial z}{\partial L / \partial t}\right]>0, \forall z \in \mathrm{U}, \forall t \geq 0
$$

From the definition (3.16), for all $t \geq 0$ we have

$$
\begin{aligned}
\frac{|L(z ; t)|}{\left|a_{1}(t)\right|} & \leq\left|\frac{(\delta-1)}{\delta+t}\right||G(z)|+\left|\frac{(1+t)}{\delta+t}\right|\left|z G^{\prime}(z)\right| \\
& \leq|G(z)|+\left|z G^{\prime}(z)\right| \leq \frac{r}{(1-r)}+\frac{r}{(1-r)^{2}} \\
& \leq \frac{r}{(1-r)^{2}} \quad(|z| \leq r<1 ; t \geq 0)
\end{aligned}
$$

hence the second assumption of Lemma 1 holds, and according to this lemma we conclude that the function $L(z ; t)$ is a subordination chain.

Now, by using Lemma 5, we will show that $F(z) \prec G(z)$. Without loss of generality, we can assume that $F$ and $G$ are analytic and univalent in $\bar{U}$ and $G^{\prime}(\zeta) \neq 0$ for $|\zeta|=1$. If not, then we could replace $F$ with $F_{r}(z)=F(r z)$ and $G$ with $G_{r}(z)=G(r z)$, where $r \in(0,1)$. These new functions will have the desired properties and we would prove our result using part (iii) of Lemma 5.

With our above assumption, we will use part (i) of the Lemma 5. If we denote by $\psi\left(G(z), z G^{\prime}(z)\right)=\varphi(z)$, we only need to show that $\psi \in \Psi[\varphi, G]$, i.e. $\psi$ is an admissible function. Because

$$
\psi\left(G(\zeta), m \zeta G^{\prime}(\zeta)\right)=\left(1-\frac{1}{\delta}\right) G(z)+\frac{(1+t)}{\delta} z G^{\prime}(z)=L(\zeta ; t)
$$

where $m=1+t, t \geq 0$, since $L(z ; t)$ is a subordination chain and $\varphi(z)=L(z ; 0)$, it follows that

$$
\psi\left(G(\zeta), m \zeta G^{\prime}(\zeta)\right) \notin \varphi(\mathrm{U})
$$

According to the Remark 1 we have $\psi \in \Psi[\varphi, G]$, and using Lemma 5 we obtain that $F(z) \prec G(z)$ and, moreover, $G$ is the best dominant.

In view of (1.6) taking $a=c$ and $m=0$, respectively, in Theorem 1 and using identities (1.7) and (1.8), we obtain the following results.

Corollary 1. Let for $g \in \Sigma$

$$
\begin{aligned}
& \varphi_{1}(z) \\
& =z^{2}\left\{\left(1-\beta\left(1-\frac{\ell}{\lambda} \frac{A}{a-A}\right)\right) L^{m+1}(\lambda, \ell) g(z)+\beta\left(1-\frac{\ell}{\lambda} \frac{A}{a-A}\right) L^{m}(\lambda, \ell) g(z)\right\},
\end{aligned}
$$

satisfy

$$
\begin{equation*}
\operatorname{Re}\left[1+\frac{z \varphi_{1}^{\prime \prime}(z)}{\varphi_{1}^{\prime}(z)}\right]>-\rho(z \in \mathrm{U}) \tag{3.19}
\end{equation*}
$$

where, $\rho=0$ if $\operatorname{Re}(\boldsymbol{\delta})=1$ and for $\operatorname{Re}(\boldsymbol{\delta})>1, \rho$ is given by (3.3) and (3.4). If $f \in \Sigma$ such that

$$
z^{2}\left\{\left(1-\beta\left(1-\frac{\ell}{\lambda} \frac{A}{a-A}\right)\right) L^{m+1}(\lambda, \ell) f(z)+\beta\left(1-\frac{\ell}{\lambda} \frac{A}{a-A}\right) L^{m}(\lambda, \ell) f(z)\right\} \prec \varphi_{1}(z),
$$

then

$$
z^{2} L^{m}(\lambda, \ell) f(z) \prec z^{2} L^{m}(\lambda, \ell) g(z) \quad(z \in U),
$$

and the function $z^{2} L^{m}(\lambda, \ell) g(z)$ is the best dominant.
Corollary 2. Let for $g \in \Sigma$

$$
\varphi_{2}(z)=z^{2}\left\{(1-\beta)\left(1-\frac{\lambda}{\ell} \frac{a-A}{A}\right) \tilde{I}_{A}^{a, c} g(z)+\left[(1-\beta)\left(\frac{\lambda}{\ell} \frac{a-A}{A}\right)+\beta\right] \bar{I}_{A}^{a+1, c} g(z)\right\},
$$

satisfy

$$
\begin{equation*}
\operatorname{Re}\left[1+\frac{z \varphi_{2}^{\prime \prime}(z)}{\varphi_{2}^{\prime}(z)}\right]>-\rho(z \in \mathrm{U}) \tag{3.20}
\end{equation*}
$$

where, $\rho=0$ if $\operatorname{Re}(\boldsymbol{\delta})=1$ and for $\operatorname{Re}(\delta)>1, \rho$ is given by (3.3) and (3.4). If $f \in \Sigma$ such that

$$
z^{2}\left\{(1-\beta)\left(1-\frac{\lambda}{\ell} \frac{a-A}{A}\right) \tilde{I}_{A}^{a, c} f(z)+\left[(1-\beta)\left(\frac{\lambda}{\ell} \frac{a-A}{A}\right)+\beta\right] \widehat{I}_{A}^{a+1, c} f(z)\right\} \prec \varphi_{2}(z),
$$

then

$$
z^{2} \tilde{I}_{A}^{a, c} f(z) \prec z^{2} I_{A}^{a, c} g(z)
$$

and the function $z^{2} \mathcal{I}_{A}^{a, c} g(z)$ is the best dominant.
Also, if we put $\beta=0$ and $\beta=1$ in Corollaries 1 and 2, we obtain the following results.

Corollary 3. Let $f, g \in \Sigma$, the operator $L^{m}(\lambda, \ell)$ defined by (1.3). Also, let

$$
\operatorname{Re}\left[1+\frac{z \psi_{1}^{\prime \prime}(z)}{\psi_{1}^{\prime}(z)}\right]>-\sigma_{1} \quad(z \in \mathrm{U})
$$

and

$$
\psi_{1}=z^{2} L^{m+1}(\lambda, \ell) g(z),
$$

where, $\sigma_{1}=0$ if $\frac{\ell}{\lambda}=1$ and for $\frac{\ell}{\lambda}>1$

$$
\sigma_{1} \leq\left\{\begin{array}{lr}
\frac{\ell-\lambda}{2 \lambda} & 1<\frac{\ell}{\lambda}<2, \\
\frac{\lambda}{2(\ell-\lambda)} & \frac{\ell}{\lambda}>2,
\end{array}\right.
$$

then

$$
\begin{aligned}
& z^{2} L^{m+1}(\lambda, \ell) f(z) \prec z^{2} L^{m+1}(\lambda, \ell) g(z) \\
& \Rightarrow z^{2} L^{m}(\lambda, \ell) f(z) \prec z^{2} L^{m}(\lambda, \ell) g(z),
\end{aligned}
$$

and the function $z^{2} L^{m}(\lambda, \ell) g(z)$ is the best dominant.
Corollary 4. Let $f, g \in \Sigma$, the operator $\widetilde{I}_{A}^{a, c}$ defined by
(1.4). Also, let

$$
\operatorname{Re}\left[1+\frac{z \psi_{2}^{\prime \prime}(z)}{\psi_{2}^{\prime}(z)}\right]>-\sigma_{2}(z \in \mathrm{U})
$$

where, $\sigma_{2}=0$ if $\operatorname{Re}\left(\frac{a}{A}\right)=2$ and for $\operatorname{Re}\left(\frac{a}{A}\right)>2$,

$$
\sigma_{2} \leq\left\{\begin{array}{lr}
\frac{\operatorname{Re}\left(\frac{a}{A}\right)-2}{2} & 2<\operatorname{Re}\left(\frac{a}{A}\right)<3,  \tag{3.21}\\
\frac{1}{2\left(\operatorname{Re}\left(\frac{a}{A}\right)-2\right)} & \operatorname{Re}\left(\frac{a}{A}\right)>3,
\end{array}\right.
$$

and

$$
\begin{equation*}
\left(\operatorname{Im}\left(\frac{a}{A}\right)\right)^{2} \leq\left(\operatorname{Re}\left(\frac{a}{A}\right)-2-2 \sigma_{2}\right)\left(\frac{1}{2 \sigma_{2}}-\operatorname{Re}\left(\frac{a}{A}\right)+2\right) . \tag{3.22}
\end{equation*}
$$

The equality in (3.21) and (3.22) occur only when $\operatorname{Im}(\delta)=0$. then

$$
z^{2} \widetilde{I}_{A}^{a+1, c} f(z) \prec z^{2} \widehat{I}_{A}^{a+1, c} g(z) \Rightarrow z^{2} \widehat{I}_{A}^{a, c} f(z) \prec z^{2} \widehat{I}_{A}^{a, c} g(z),
$$

and the function $z^{2} \overparen{I}_{A}^{a, c} g(z)$ is the best dominant.
Next, we will give an interesting special case of our main results, obtained for an appropriate choice of the function $g$.

Thus, let consider the function $g \in \Sigma$ defined by

$$
g(z)=z^{-1}+\sum_{n=0}^{\infty} a_{n} z^{n},\left(z \in U^{*}\right)
$$

with

$$
\begin{gathered}
a_{n}=\frac{1}{n+2}\left(1+(n+1)\left[(1-\beta)\left(\frac{\lambda}{\ell}\right)+\beta\left(\frac{A}{a-A}\right)\right]\right)^{-1} \\
\cdot\left(\frac{\ell}{\ell+\lambda(n+1)}\right)^{m} \frac{\Gamma(a-A)}{\Gamma(c-A)} \frac{\Gamma(c+n A)}{\Gamma(a+n A)}\binom{-2(\rho+1)}{n+1}, n \geq 0,
\end{gathered}
$$

where $\rho$ is given by (3.3), and

$$
\binom{\tau}{n}=\frac{\tau(\tau-1) \ldots(\tau-n+1)}{n!},(\tau \in C, n \in N) .
$$

If the function $\varphi$ is defined by (3.2), then

$$
\varphi(z)=\frac{1-(1+z)^{-(2 \rho+1)}}{2 \rho+1},(z \in \mathrm{U})
$$

where the power is the principal one, i.e.

$$
\left.(1+z)^{-(2 \rho+1)}\right|_{z=0}=1
$$

A simple computation shows that

$$
\operatorname{Re}\left[1+\frac{z \varphi^{\prime \prime}(z)}{\varphi^{\prime}(z)}\right]=\operatorname{Re} \frac{1-(2 \rho+1) z}{1+z}>-\rho,(z \in \mathrm{U}),
$$

and from Theorem 1, we obtain:
Example 1. Let $\rho$ be given by (3.3). If $f \in \Sigma$ such that

$$
\begin{aligned}
& z^{2}\left\{(1-\beta) L_{\lambda, \ell}^{m+1}(a, c, A) f(z)+\beta L_{\lambda, \ell}^{m}(a+1, c, A) f(z)\right\} \\
& \prec \frac{1-(1+z)^{-(2 \rho+1)}}{2 \rho+1},
\end{aligned}
$$

then

$$
\begin{aligned}
& z^{2} L_{\lambda, \ell}^{m}(a, c, A) f(z) \\
& \prec z+\sum_{n=0}^{\infty} \frac{1}{n+2}\left(1+(n+1)\left[(1-\beta)\left(\frac{\lambda}{\ell}\right)+\beta\left(\frac{A}{a-A}\right)\right]\right)^{-1}\binom{-2(\rho+1)}{n+1} z^{n+2},
\end{aligned}
$$

and the right-hand side function is the best dominant.
Also, let consider the function $g \in \Sigma$ defined by

$$
g(z)=z^{-1}+\sum_{n=0}^{\infty} a_{n} z^{n},\left(z \in U^{*}\right)
$$

with

$$
\begin{aligned}
a_{n}= & \frac{1}{n+2}\left(1+(n+1)\left[(1-\beta)\left(\frac{\lambda}{\ell}\right)+\beta\left(\frac{A}{a-A}\right)\right]\right)^{-1} . \\
& \cdot\left(\frac{\ell}{\ell+\lambda(n+1)}\right)^{m} \frac{\Gamma(a-A)}{\Gamma(c-A)} \frac{\Gamma(c+n A)}{\Gamma(a+n A)}\binom{-2(\rho+1)}{n+1},(n \geq 0),
\end{aligned}
$$

where $\rho$ is given by (3.3). If the function $\varphi_{1}$ is defined by (3.19), then

$$
\varphi_{1}(z)=\frac{1-(1+z)^{-(2 \rho+1)}}{2 \rho+1},(z \in \mathrm{U}),
$$

where the power is the principal one, i.e.

$$
\left.(1+z)^{-(2 \rho+1)}\right|_{z=0}=1 .
$$

A simple computation shows that

$$
\operatorname{Re}\left[1+\frac{z \varphi_{1}^{\prime \prime}(z)}{\varphi_{1}^{\prime}(z)}\right]=\operatorname{Re} \frac{1-(2 \rho+1) z}{1+z}>-\rho,(z \in \mathrm{U})
$$

and from Corollary 1, we obtain:
Example 2. Let $\rho$ be given by (3.3). If $f \in \Sigma$ such that

$$
\begin{aligned}
& z^{2}\left\{\left(1-\beta\left(1-\frac{\ell}{\lambda} \frac{A}{a-4}\right)\right) L^{m+1}(\lambda, \ell) f(z)+\beta\left(1-\frac{\ell}{\lambda} \frac{A}{a-A}\right) L^{m}(\lambda, \ell) f(z)\right\} \\
& \prec \frac{1-(1+z)^{-(2 \rho+1)}}{2 \rho+1},
\end{aligned}
$$

then

$$
\begin{aligned}
& z^{2} L_{\lambda, \ell}^{m}(a, c, A) f(z) \\
& \prec z+\sum_{n=0}^{\infty} \frac{1}{n+2}\left(1+(n+1)\left[(1-\beta)\left(\frac{\lambda}{\ell}\right)+\beta\left(\frac{A}{a-A}\right)\right]\right)^{-1}\binom{-2(\rho+1)}{n+1} z^{n+2},
\end{aligned}
$$

and the right-hand side function is the best dominant.

Finally, let consider the function $g \in \Sigma$ defined by

$$
g(z)=z^{-1}+\sum_{n=0}^{\infty} a_{n} z^{n},\left(z \in U^{*}\right)
$$

with

$$
\begin{gathered}
a_{n}=\frac{1}{n+2}\left(1+(n+1)\left[(1-\beta)\left(\frac{\lambda}{\ell}\right)+\beta\left(\frac{A}{a-A}\right)\right]\right)^{-1} . \\
\left(\frac{\ell}{\ell+\lambda(n+1)}\right)^{m} \frac{\Gamma(a-A)}{\Gamma(c-A)} \frac{\Gamma(c+n A)}{\Gamma(a+n A)}\binom{-2(\rho+1)}{n+1},(n \geq 0),
\end{gathered}
$$

where $\rho$ is given by (3.3), and

$$
\binom{\tau}{n}=\frac{\tau(\tau-1) \ldots(\tau-n+1)}{n!},(\tau \in C, n \in N) .
$$

If the function $\varphi_{2}$ is defined by (3.20), then

$$
\varphi_{2}(z)=\frac{1-(1+z)^{-(2 \rho+1)}}{2 \rho+1},(z \in \mathrm{U}),
$$

where the power is the principal one, i.e.

$$
\left.(1+z)^{-(2 \rho+1)}\right|_{z=0}=1 .
$$

A simple computation shows that

$$
\operatorname{Re}\left[1+\frac{z \varphi_{2}^{\prime \prime}(z)}{\varphi_{2}^{\prime}(z)}\right]=\operatorname{Re} \frac{1-(2 \rho+1) z}{1+z}>-\rho,(z \in \mathrm{U})
$$

and from Corollary 2, we obtain:
Example 3. Let $\rho$ be given by (3.3). If $f \in \Sigma$ such that

$$
\begin{aligned}
& z^{2}\left\{(1-\beta)\left(1-\frac{\lambda}{\ell} \frac{a-A}{A}\right) \tilde{I}_{A}^{a, c} f(z)+\left[(1-\beta)\left(\frac{\lambda}{\ell} \frac{a-A}{A}\right)+\beta\right] \widetilde{I}_{A}^{a+1, c} f(z)\right\} \\
& \prec \frac{1-(1+z)^{-(2 \rho+1)}}{2 \rho+1},
\end{aligned}
$$

then

$$
\begin{aligned}
& z^{2} L_{\lambda, \ell}^{m}(a, c, A) f(z) \\
& \prec z+\sum_{n=0}^{\infty} \frac{1}{n+2}\left(1+(n+1)\left[(1-\beta)\left(\frac{\lambda}{\ell}\right)+\beta\left(\frac{A}{a-A}\right)\right]\right)^{-1}\binom{-2(\rho+1)}{n+1} z^{n+2},
\end{aligned}
$$

and the right-hand side function is the best dominant.

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