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Nonlinear algebraic systems of equations with many variables

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Abstract

In this work the criterion for the existence of the common eigenvalues of the several operator pencils in Hilbert spaces is proved. The author gives the new manner for the study of nonlinear algebraic system of equations, when the number of equations is more or equal to the number of variables. For the studying of this problem the author uses essentially the criterion of the existence of the common solutions of the nonlinear several algebraic equations with many variables. With the help of the notion of the resultant for two polynomials the existence and the number of the common solutions of the nonlinear several algebraic systems of equations with many variables are determined.

Keywords

Eigenvalue, Algebraic Equations, Resultant, Tensor Product, Space, Criterion

1. Introduction

This paper is devoted to a study of nonlinear algebraic systems with $n(n \geq 2)$ unknown variables, that is a continuation of the author's research has been published in [2], [3], [4]. Previously, for a nonlinear algebraic system was built the analogue of the determinant of Cramer [4] and was obtained the necessary and sufficient conditions for the existence of solutions of nonlinear algebraic equations with a complex dependence on two or three variables. In [3] for the nonlinear algebraic system with two variables and polynomial dependence on them the author defines the number of the solutions under the some conditions on the coefficients of this system.

Definition 1.1.

Let

$$A(\lambda) = A_0 + \lambda A_1 + \lambda^2 A_2 + \ldots + \lambda^n A_n, \\ B(\lambda) = B_0 + \lambda B_1 + \lambda^2 B_2 + \ldots + \lambda^m B_m \tag{1}$$

be two operator pencils depending on the same parameter λ and acting, generally speaking, in different Hilbert spaces H_1, H_2, correspondingly.

Resultant of two operator pencils $A(\lambda)$ and $B(\lambda)$ is the operator, presented by the determinant (2) and acting in the space $(H_1 \otimes H_2)^{n+m}$ - direct sum of $n+m$ copies of tensor product $H_1 \otimes H_2$ of spaces H_1 and H_2.

$$\operatorname{Re}s(A(\lambda), B(\lambda)) = \begin{pmatrix} A_0 \otimes E_2 & A_1 \otimes E_2 & \ldots & A_n \otimes E_2 & \ldots & 0 \\ . & . & \ldots & . & \ldots & . \\ 0 & 0 & \ldots A_0 \otimes E_2 & A_1 \otimes E_2 & \ldots & A_n \otimes E_2 \\ E_1 \otimes B_0 & E_1 \otimes B_1 & \ldots & E_1 \otimes B_m & .. & 0 \\ . & . & \ldots & . & \ldots & . \\ . & . & \ldots E_1 \otimes B_0 & E_1 \otimes B_1 & \ldots. & E_1 \otimes B_m \end{pmatrix} \tag{2}$$

In a matrix $\operatorname{Re}s(A(\lambda), B(\lambda))$ the number of rows with operators A_i is equal to leading degree of parameter λ in the operator pencil $B(\lambda)$, that is m; the number of rows in matrix $\operatorname{Re}s(A(\lambda), B(\lambda))$ with operators B_i coincides with the leading degree of parameter λ in the operator pencil $A(\lambda)$, that is n. One major application of the matrix theory is calculation of determinants. It turns out that a mapping is invertible if and only if the determinant of this matrix is not zero.

The concept of an abstract analogue of the resultant of two polynomial pencils when they have the same degree with respect to the parameter is given in the work of Khaynig [6], and in the case of different degrees of parameter an abstract analogue of resultant is studied by Balinskii [1].

The operator (2) is the generalization of the notion of resultant

$$\operatorname{Re}s(f,g)=\begin{pmatrix} a_n & a_{n-1} & \dots & a_1 & a_0 & 0 & \dots & \dots & 0 & 0 \\ 0 & a_n & \dots & a_2 & a_1 & a_0 & \dots & \dots & 0 & 0 \\ . & . & \dots & . & . & . & \dots & \dots & . & . \\ . & . & \dots & . & . & . & \dots & \dots & . & . \\ 0 & 0 & \dots & 0 & 0 & a_n & \dots & \dots & a_1 & a_0 \\ b_m & b_{m-1} & \dots & b_3 & b_2 & b_1 & b_0 & \dots & 0 & 0 \\ 0 & b_m & \dots & b_4 & b_3 & b_2 & b_1 & b_0\dots & 0 & 0 \\ & . & \dots & . & . & . & . & \dots & . & . \\ & . & \dots & . & . & . & . & \dots & . & . \\ & 0 & 0 & 0 & 0 & b_m & b_{m-1} & \dots & b_1 & b_0 \end{pmatrix} \quad (3)$$

constructed for polynomials

$$f(x)=a_nx^n+a_{n-1}x^{n-1}+\dots+a_0,\quad a_n\neq 0;$$
$$g(x)=b_mx^m+b_{m-1}x^{m-1}+\dots+b_0,\quad b_m\neq 0;$$

Definition 1.2.

The operator $B^+_{s,i}$ is induced by an operator $B_{s,i}$, acting from the space H_i, to the tensor space $H=H_1\otimes\dots\otimes H_n$, if on each decomposable tensor $x=x_1\otimes\dots\otimes x_n$ of tensor product space $H=H_1\otimes\dots\otimes H_n$ we have $B^+_{s,i}x=x_1\otimes\dots\otimes x_{i-1}\otimes B_{s,i}x_i\otimes x_{i+1}\otimes\dots\otimes x_n$,and on all the other elements of $H=H_1\otimes\dots\otimes H_n$ the operator $B^+_{s,i}$ is defined on linearity and continuity.

2. Criterion of Existence the Common Point of Spectra of Two Operator Pencils

Let all operators $A_i\ (i=0,1,\dots,n)$ (correspondingly, $B_i\ (i=0,1,\dots,m)$) are bounded in the Hilbert space (correspondingly, H_2). Operator A_n or B_m has bounded inverse. By [1] the existence of non-zero kernel of the operator $\operatorname{Re}s(A(\lambda),B(\lambda))$ is the necessary and sufficient condition for the existing of the common point of spectra of operators $A(\lambda)$ and $B(\lambda)$. If the spectrum of each operator $A(\lambda)$ and $B(\lambda)$ contains only eigenvalues, then a common point of spectra of these operators is their common eigenvalue.

3. Criterion of Existence the Common Point of Spectra of the Several Operator Pencils

We use the result from [2]. Let the operator pencils

$$\{B_i(\lambda)=B_{0,i}+\lambda B_{1,i}+\dots+\lambda^{k_i}B_{k_i,i},\quad i=1,2,\dots,n \quad (4)$$

depend on the same parameter λ , $B_i(\lambda)$ are the operator pencils, acting in finite-dimensional Hilbert space H_i , correspondingly. Without loss of generality we assume $k_1\geq k_2\geq\dots\geq k_n$.

In the space $H^{k_1+k_2}$ (the direct sum of k_1+k_2 tensor product $H=H_1\otimes\dots\otimes H_n$ of spaces $H_1,H_2,\dots,H_n$) we introduce the operators $R_i\ (i=1,\dots,n-1)$ by means of an operator matrix

$$R_{i-1}=\begin{pmatrix} B^+_{0,1} & B^+_{1,1} & \cdots\cdots & B^+_{k_1,1} & \cdots\cdots\cdot & 0 \\ 0 & B^+_{0,1} & B^+_{1,1}\cdot\cdot & B^+_{k_1-1.1} & B^+_{k_1,1}\cdot\cdot & 0 \\ . & . & \cdots\cdot & . & \cdots\cdots & . \\ 0 & 0 & \cdot\cdot B^+_{0,1} & B^+_{1,1} & \cdots\cdots & B^+_{k_1,1} \\ B^+_{0,i} & B^+_{1,i} & \cdots\cdots & B^+_{k_i,i} & 0\cdots\cdot & 0 \\ 0 & B^+_{0,i} & B^+_{1,i}\cdots & \cdot & B^+_{k_i,i}\cdot\cdot & 0 \\ . & . & \cdots\cdots & . & \cdots\cdots & . \\ 0 & 0 & \cdots B^+_{0,i} & B^+_{1,i} & \cdots\cdots & B^+_{k_i,i} \end{pmatrix},$$

$$i=2,\dots,n. \quad (5)$$

Let the operator pencil $B_i(\lambda)\,(i=1,2,\dots,n)$ acts in a finite dimensional Hilbert space $H_i\ (i=1,\dots,n)$, correspondingly.

The number of rows with the operators $B^+_{s,1}, s=0,1,\dots,k_1$ in the matrix operators R_{i-1} is equal to k_2, a number of rows with the operators $B_{s,i}, s=0,1,\dots,k_i$ is equal to k_1. We denote $\sigma_p(B_i(\lambda))$ the set of eigenvalues of $B_i(\lambda)$.

Theorem 1. ([2], [4]). Let all the operators $B_{s,i}$ in (4) are bounded and the operator $B_{k_1,1}$ has inverse. Then $\bigcap_{i=1}^{n}\sigma_p(B_i(\lambda))\neq\{\theta\}$ if and only if

$$\bigcap_{i=1}^{n-1}KerR_i\neq\{\theta\},\ (KerB_{k_1}=\{\theta\}). \quad (6)$$

Proof of Theorem 1. Necessity. We suppose, that pencils $B_i(\lambda)$ have a common eigenvalue λ^0 . For everyone i there are such elements $x_i\in H_i$, that $B^+_i(\lambda^0)x_1\otimes\dots\otimes x_n=0,\quad i=1,2,\dots,n.$

It is not difficult to see, if the element

$$X=\left(x_1\otimes\ldots\otimes x_n,\ \lambda^0 x_1\otimes\ldots\otimes x_n,\ldots,\left(\lambda^0\right)^{k_1+k_2-1}x_1\otimes\ldots\otimes x_n\right)$$

enters the kernel of an operator R_i for each $i=1,2,\ldots,n-1$, then $X\in\bigcap_{i=1}^{n-1}KerR_i$.

Sufficiency. Let $\bigcap_{i=1}^{n-1}KerR_i\neq\{\theta\}$ and an element $X=(\tilde{x}_1,\tilde{x}_2,\ldots,\tilde{x}_{k_1+k_2})\in\bigcap_{i=1}^{n-1}KerR_i,\quad \tilde{x}_i\in H$. Then element X is in the kernels of operators $R_1,R_2,\ldots,R_{n-1}$, i. e. $R_s(\tilde{x}_1,\tilde{x}_2,\ldots,\tilde{x}_{k_1+k_2})=\theta;\ s=1,2,\ldots,n-1$. Expression $\bigcap_{i=1}^{n-1}KerR_i\neq\{\theta\}$ means that there is such nonzero element $\left(\sum_{i=1}^{s}x_{1,i}^{(s)}\otimes x_{2,i}^{(s)}\otimes\ldots\otimes x_{n,i}^{(s)}\right)_{s=1}^{k_1+k_2}\in(H_1\otimes\ldots\otimes H_n)^{k_1+k_2}$ that the equalities

$$\begin{gathered}
B_{0,1}^{+}\sum_{i=1}^{s_1}x_{1,i}^{(1)}\otimes x_{2,i}^{(1)}\otimes\ldots\otimes x_{n,i}^{(1)}+\ldots+B_{k_1,1}^{+}\sum_{i=1}^{s_{k_1+1}}x_{1,i}^{(k_1+1)}\otimes\ldots\otimes x_{n,i}^{(k_1+1)}=0\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
B_{0,1}^{+}\sum_{i=1}^{s_{k_2}}x_{1,i}^{(k_2)}\otimes x_{2,i}^{(k_2)}\otimes\ldots\otimes x_{n,i}^{(k_2)}+\ldots+B_{k_1,1}^{+}\sum_{i=1}^{s_{k_1+k_2}}x_{1,i}^{(k_1+k_2)}\otimes\ldots\otimes x_{n,i}^{(k_1+k_2)}=0\\
B_{0,i}^{+}\sum_{i=1}^{s_1}x_{1,i}^{(1)}\otimes x_{2,i}^{(1)}\otimes\ldots\otimes x_{n,i}^{(1)}+\ldots+B_{k_i,i}^{+}\sum_{i=1}^{s_{k_2+1}}x_{1,i}^{(k_i+1)}\otimes\ldots\otimes x_{n,i}^{(k_i+1)}=0\\
B_{0,i}^{+}\sum_{i=1}^{s_{k_1}}x_{1,i}^{(k_1)}\otimes\ldots\otimes x_{n,i}^{(k_1)}+\ldots+B_{k_i,i}^{+}\sum_{i=1}^{s_{k_1+k_2}}x_{1,i}^{(k_1+k_2)}\otimes\ldots\otimes x_{n,i}^{(k_1+k_2)}=0\\
i=2,\ldots n
\end{gathered}\tag{7}$$

satisfy. Then an element $\left(\sum_{i=1}^{\infty}x_{1,i}^{(s)}\otimes x_{2,i}^{(s)}\otimes\ldots\otimes x_{n,i}^{(s)}\right)_{s=1}^{k_1+k_2}$ enters the kernel of Resultant of operators $B_1(\lambda)$ and $B^{++}(\lambda,\alpha_2,\ldots,\alpha_n)=\alpha_2B_2^{++}(\lambda)+\alpha_3B_3^{++}(\lambda)+\ldots+\alpha_nB_n^{++}(\lambda)$. $B^{++}(\lambda,\alpha_2,\ldots,\alpha_n)$ is the operator induced to the space $H_2\otimes\ldots\otimes H_n$ by the operator $B_i(\lambda)$ and $\alpha_i\ (i=2,3,\ldots,n)$ are arbitrary complex numbers. The resultant of pencils $B_1(\lambda)$ and $B^{++}(\lambda,\alpha_2,\ldots,\alpha_n)$ acts in the direct sum of k_1+k_2 copies of tensor product $H_1\otimes\ldots\otimes H_n$ of the spaces $H_1,\ldots,H_n$.

Further we use the known property of the elements of tensor product space. It is known that the representation of the element in tensor product space is not unique. For each element of the tensor product space there is the number coinciding with the minimal number of decomposable tensors, necessary for the representation of this element. This number is named the rank of element. If the sum decomposable tensors in the presentation of element are more than the rank of this element then one-nominal components of the given element are linear dependent. Having transferred to each line of equalities (7) one decomposable tensor from left side in the right side of the corresponding equality, we get, that the series standing at the left side in each equality have a rank 1, as they are equal to a decomposable tensor, standing in the right side of this equality. Thus, between one-nominal components of all terms entering into expression (7) there is a linear dependence and an element standing at the left in all equalities in (7) is a decomposable tensor.

We have that $\left(\sum_{i=1}^{s}x_{1,i}^{(s)}\otimes x_{2,i}^{(s)}\otimes\ldots\otimes x_{n,i}^{(s)}\right)\in H_1\otimes\ldots\otimes H_n$ is decomposable vector for all values of number s. Using [2] we prove there is a number λ being the common point of spectra of operators $B_1(\lambda)$ and $B^{++}(\lambda,\alpha_2,\ldots,\alpha_n)$ at all values α_i. The last means, that λ is a common eigenvalue of all pencils $B_i(\lambda)$, $i=1,2,\ldots,n$. Theorem 1 is proved.

4. The Nonlinear Algebraic System of Equations with Many Variables

Consider the system

$$\begin{gathered}
a_i(x,y,z_1,\ldots,z_k)=a_{0i}+a_{1i}x+\ldots+a_{m_i,i}x^{m_i}+\\
+a_{m_i+1,i}y+\ldots+a_{m_i+n_i,i}y^{n_i}+\sum_{r=1}^{k}a_{m_i+n_i+r,i}z_r=0\\
i=1,\ldots,k+2
\end{gathered}\tag{8}$$

The study of the algebraic system (8) is carried out under the scheme, carried out in [3] and [4]. Fixing all the variables, except for a variable z_k, we obtain the $k+2$ polynomials with one variable z_k. All these polynomials have a common solution if and only if the kernel of resultants of all pair of polynomial

$$\begin{gathered}
a_1(x,y,z_1,\ldots,z_k)=a_{01}+a_{11}x+\ldots+a_{m_1,1}x^{m_1}+\\
+a_{m_1+1,1}y+\ldots+a_{m_1+n_1,1}y^{n_1}+\sum_{r=1}^{k}a_{m_1+n_1+r,1}z_r=0\\
a_i(x,y,z_1,\ldots,z_k)=a_{0i}+a_{1i}x+\ldots+a_{m_i,i}x^{m_i}+\\
+a_{m_i+1,i}y+\ldots+a_{m_i+n_i,i}y^{n_i}+\sum_{r=1}^{k}a_{m_i+n_i+r,i}z_r=0\\
i=2,\ldots,k+2
\end{gathered}\tag{9}$$

has a nonzero intersection.

Then, for each i there is nonzero number z_k, that is the common solution of pair of polynomials from (9) at all meanings $i=2,\ldots,k+2$, naturally, when all the other variables (except for a variable z_k) are fixed.

We introduce the following notations:

$$\begin{gathered}
\tilde{a}_i(x,y,z_1,\ldots,z_k)=a_{0i}+a_{1i}x+\ldots+\\
+a_{m_i,i}x^{m_i}+a_{m_i+1,i}y+\ldots+a_{m_i+n_i,i}y^{n_i}+\sum_{r=1}^{k-1}a_{m_i+n_i+r}z_r\\
i=1,2,\ldots,k+2
\end{gathered}$$

$$\tilde{a}_i(x,y,z_1,...,z_k) = a_{0i} + a_{1i}x + ... + \\ + a_{m_i,i}x^{m_i} + a_{m_i+1,i}y + ... + a_{m_i+n_i,i}y^{n_i} + \sum_{r=1}^{k-1} a_{m_i+n_i+r}z_r \quad (10) \\ i = 1,2,...,k+2$$

Let the determinants of resultants

$$\mathrm{Re}\,s(a_1(z_k), a_i(z_k)) = \begin{pmatrix} \tilde{a}_1(x,y,z_1,...,z_{k-1}) & a_{m_1+n_1+k,1} \\ \tilde{a}_i(x,y,z_1,...,z_{k-1}) & a_{m_i+n_i+k,i} \end{pmatrix}$$

is equal to zero for all values $i=2,3,...,k+2$. If we consider the decomposition of these determinants then we come to the nonlinear algebraic system with $k+1$ variables and $k+1$ equations.

The resulting algebraic system with variables $x, y.z_1,..., z_{k-1}$ has the form

$$\tilde{a}_1(x,y,z_1,...,z_{k-1})a_{m_i+n_i+k,i} - \\ - a_{m_1+n_1+k,1}\tilde{a}_i(x,y,z_1,...,z_{k-1}) = 0 \quad (11) \\ i = 2,...,k+2$$

The existence the system with S variables is a necessary and sufficient condition for the existence of solutions of the nonlinear algebraic system with $s+1$ variables. Thus, we arrive at the existence of a solution of the original nonlinear algebraic system (8).

The number of solutions of all intermediate algebraic systems, obtained in the course of the proof, coincides with the number of solutions of the original algebraic system (8).

Consider the resulting procedures conducted nonlinear algebraic system of equations in two variables.

Let obtained algebraic system in two variables has the form

$$a(x,y) = a_0 + a_1x + ... + a_{m_1}x^{m_1} + a_{m_1+1}y + ... + a_{m_1+n_1}y^{n_1} = 0$$
$$b(x,y) = b_0 + b_1x + ... + b_{m_2}x^{m_2} + b_{m_2+1}y + ... + b_{m_2+n_2}y^{n_2} = 0 \quad (12)$$

Previous arguments show that the greatest degree of variables x, y in the systems (8) and (12) are the same.

Each of the equation in (11) is the decomposition of the determinant of the resultant $\mathrm{Re}\,s(a_1(z_k),a_i(z_k)) = \begin{pmatrix} \tilde{a}_1(x,y,z_1,...,z_{k-1}) & a_{m_1+n_1+k,1} \\ \tilde{a}_i(x,y,z_1,...,z_{k-1}) & a_{m_i+n_i+k,i} \end{pmatrix}$ and (11) means that system of equations has a solution $z_{k,i}(x,y,z_1,...,z_{k-1}),\ i = 2,...,k+2$, respectively. If the $k+1$ variables $(x,y,z_1,...,z_{k-1})$ are the solution of (11) at each value $i = 1,2,...,k+1$ then the $z_{k,i}(x,y,z_1,...,z_{k-1}), i = 2,...,k+1$ is the solution of the first equation of the system (8). It is also clear that if the equation $a_1(x,y,z_1,...,z_k) = 0$ has a solution $(x,y,z_1,...,z_{k-1})$ then the system (11) has a unique solution $z_k(x,y,z_1,...,z_{k-1})$, since the variable z_k enters the equation $a_1(x,y,z_1,...,z_k) = 0$ linearly. In addition, the $z_k(x,y,z_1,...,z_{k-1})$ is the of common solution of all other equations of the system (11). Therefore, $z_{k,i}(x,y,z_1,...,z_{k-1}), i = 2,...,k+2$ are equal to $z_k(x,y,z_1,...,z_{k-1})$ and to each solution of (11) corresponds to only one solution of (8). The system (11) and the system (8) have an equal number of solutions. We continue this process. Every time we come to the algebraic systems with the number of unknowns, one less. The final equation in one variable is the decomposition of the determinant of the resultant of two polynomials depending on a single variable (the other variables are fixed). In addition, each solution of a last polynomial generates the final solution of the nonlinear algebraic system with two variables.

Further, each equation of nonlinear algebraic system of two variables is the decomposition of the determinants of resultants, built for an algebraic system of three variables.

The existence of solutions of the system with k variables is the necessary and sufficient condition for the existence of solutions of the previous system, the number of variables in which one more.

Naturally, the establishment of the existence of solutions of the algebraic system (12) is possible through the use of the results of multiparameter system of operators with two parameters in a finite dimentional space.

In the case of our system (12) the number of solutions is defined under the conditions on the coefficients of the system (12). Let us give an independent proof of determining the number of solutions of the system (12) without the involvement of the methods of the multiparameter systems of operators, as research technique of multiparameter systems may be unfamiliar to many readers.

For our purpose, we use the concept of the resultant of two polynomials in the same variable. We fix a variable x in (12).

Introduce the notations

$$\tilde{a}(x_0) = a_0 + a_1x_0 + ... + a_{m_1}x_0^{m_1}, \quad \tilde{b}(x_0) = b_0 + b_1x_0 + ... + b_{m_2}x_0^{m_2}$$

We have two polynomials in the same variable y.

$$a(x_0, y) = \tilde{a}(x_0) + a_{m_1+1}y + ... + a_{m_1+n_1}y^{n_1}$$

$$b(x_0, y) = \tilde{b}(x_0) + b_{m_2+1}y + ... + b_{m_2+n_2}y^{n_2}$$

Resultant of these polynomials is the operator, defined by the matrix (13).

$$\mathrm{Re}\,s(a(x_0,y),b(x_0,y)) = \begin{pmatrix} \tilde{a}(x_0) & a_{m_1+1} & \cdots & a_{m_1+n_1} & \cdots & 0 \\ . & . & \cdots & . & \cdots & . \\ 0 & 0 & \tilde{a}(x_0) & a_{m_1+1} & \cdots & a_{m_1+n_1} \\ \tilde{b}(x_0) & b_{m_2+1} & \cdots & b_{m_2+n_2} & \cdots & 0 \\ . & . & \cdots & . & \cdots & . \\ 0 & 0 & \cdots & \tilde{b}(x_0) & \cdots & b_{m_2+n_2} \end{pmatrix} \quad (13)$$

acting on the space $C^{n_1+n_2}$. In (13) the number rows with the elements a_i are repeated n_2 times, and rows with the elements b_i are repeated n_1 times. By definition, these polynomials have a common solution $y(x_0)$ if and only if the kernel of the resultant $\operatorname{Re}s(a(x_0,y),b(x_0,y))$ is not zero. We compute the determinant of this resultant. The kernel of the resultant is nonzero if and only if the determinant of the resultant is zero. Let be $\alpha=(\alpha_1,\alpha_2,...,\alpha_{n_1+n_2})\in Ker\operatorname{Re}s(a(x_0,y),b(x_0,y))$

Then the linear homogeneous system of algebraic equations

$$\begin{aligned}
&\tilde{a}(x_0)\alpha_1+a_{m_1+1}\alpha_2+...+a_{m_1+n_1}\alpha_{n_1+1}=0\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&a(x_0)\alpha_{n_2}+a_{m_1+1}\alpha_{n_2+1}+...+a_{m_1+n_1}\alpha_{n_1+n_2}\\
&\tilde{b}(x_0)\alpha_1+b_{m_2+1}\alpha_2+...+b_{m_2+n_2}\alpha_{n_2+1}=0\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\tilde{b}(x_0)a_{n_2+1}+b_{m_2+1}\alpha_{n_2+2}+...+b_{m_2+n_2}\alpha_{n_1+n_2}=0
\end{aligned}\tag{14}$$

has a solution if and only if the determinant of the resultant (13)is zero. We arrange the decomposition of the determinant in degrees of the variable x . Here we have taken into account that the variable was fixed arbitrarily.

Theorem 2. Suppose that one of the three following conditions:

a) $\max(m_1n_2,m_2n_1)=m_1n_2,\ a_{m_1}\neq 0, b_{m_2+n_2}\neq 0$;

b) $\max(m_1n_2,m_2n_1)=m_2n_1,\ b_{m_2}\neq 0$, $a_{m_1+n_1}\neq 0$

c) $m_1n_2=m_2n_1$, $a^{n_2}_{m_1+n_1}b^{n_1}_{m_2}+(-1)^{n_1n_2}a^{n_2}_{m_1}b^{n_1}_{m_2+n_2}\neq 0,$

satisfy. Then the algebraic system (12) has $\max(m_1n_2,m_2n_1)$ solutions.

Proof of the theorem2.We had that the number of solutions of all algebraic systems obtained in the course of the proof are equal, then the number of solutions of (8) not less than $\max(m_1n_2,m_2n_1)$ that required to prove. In the future, the author can apply the results of Theorem 2.

Then the author shows that these discussions on a simple example of the four algebraic equations with three unknown variables.

$$\begin{aligned}
&a_0+a_1x+a_2y+a_3z=0\\
&b_0+b_1x+b_2y+b_3z=0\\
&c_0+c_1x+c_2y+c_3z=0\\
&d_0+d_1x+d_2y+d_3z=0
\end{aligned}\tag{15}$$

Fixing x,y in the latter system, he has four equations in one variable z.. Build the resultants for the following pairs of equations.

$$\begin{aligned}
&a_0+a_1x+a_2y+a_3z=0\ ,b_0+b_1x+b_2y+b_3z=0\,;\\
&a_0+a_1x+a_2y+a_3z=0,\ c_0+c_1x+c_2y+c_3z=0\\
&a_0+a_1x+a_2y+a_3z=0,\ d_0+d_1x+d_2y+d_3z=0
\end{aligned}\tag{16}$$

The determinants of the resultants of pairs from (16) are:

$$\begin{aligned}
&b_3(a_0+a_1x+a_2y)-a_3(b_0+b_1x+b_2y)=0\\
&c_3(a_0+a_1x+a_2y)-a_3(c_0+c_1x+c_2y)=0\\
&d_3(a_0+a_1x+a_2y)-a_3(d_0+d_1x+d_3y)=0
\end{aligned}$$

or

$$\begin{aligned}
&a_0b_3-a_3b_0+(a_1b_3-a_3b_1)x+(a_2b_3-a_3b_2)y=0\\
&a_0c_3-a_3c_0+(a_1c_3-a_3c_1)x+(a_2c_3-a_3c_2)y=0\\
&a_0d_3-a_3d_0+(a_1d_3-a_3d_1)x+(a_2d_3-a_3d_2)y=0
\end{aligned}\tag{17}$$

Fix the variable x in (17). Construct the resultants of the systems of equations from (18) and (19)

$$\begin{aligned}
&a_0b_3-a_3b_0+(a_1b_3-a_3b_1)x+(a_2b_3-a_3b_2)y=0\\
&a_0c_3-a_3c_0+(a_1c_3-a_3c_1)x+(a_2c_3-a_3c_2)y=0
\end{aligned}\tag{18}$$

or

$$\begin{aligned}
&a_0b_3-a_3b_0+(a_1b_3-a_3b_1)x+(a_2b_3-a_3b_2)y=0\\
&a_0d_3-a_3d_0+(a_1d_3-a_3d_1)x+(a_2d_3-a_3d_2)y=0
\end{aligned}\tag{19}$$

Then decompositions of the determinants of the corresponding resultants of systems (18) and (19), when x is fixed, are

$$\begin{aligned}
&[(a_0b_3-a_3b_0)(a_2c_3-a_3c_2)+\\
&+(a_2b_3-a_3b_2)(a_0c_3-a_3c_0)]+\\
&+\{(a_1b_3-a_3b_1)(a_2c_3-a_3c_2)-\\
&-(a_2b_3-a_3b_2)(a_1c_3-a_3c_1)]x=0\\
&[(a_0b_3-a_3b_0)(a_2d_3-a_3d_2)-\\
&-(a_2b_3-a_3b_2)(a_0d_3-a_3d_0)]+\\
&+[(a_1b_3-a_3b_1)(a_2d_3-a_3d_2)-\\
&-(a_2b_3-a_3b_2)(a_1d_3-a_3d_1)]x=0
\end{aligned}\tag{20}$$

Construct the resultant of the last two polynomials in one variable x. So each time the existence of a solution of the obtaining system with s variables sets the existence of a solution to the proceeding system with $s+1$ variables,and the author proves: the necessary and sufficient conditions of existence the common solution of the system (20) is the fulfilling of the equality (21):

$$[(a_0b_3 - a_3b_0)(a_2c_3 - a_3c_2) + (a_2b_3 - a_3b_2)\cdot$$
$$\cdot(a_0c_3 - a_3c_0)][(a_1b_3 - a_3b_1)(a_2d_3 - a_3d_2) -$$
$$-(a_2b_3 - a_3b_2)(a_1d_3 - a_3d_1)] - [(a_1b_3 - a_3b_1)\cdot$$
$$\cdot(a_2c_3 - a_3c_2) - (a_2b_3 - a_3b_2)(a_1c_3 - a_3c_{1)}]\cdot$$

$$\cdot[(a_0b_3 - a_3b_0)(a_2d_3 - a_3d_2) - -(a_2b_3 - a_3b_2)(a_0d_3 - a_3d_0)] \neq 0 \quad (21)$$

The condition (21) is the necessary and sufficient for the existence the common solution of the four algebraic equations (15) with three variables.

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